Indian Institute of Technology Hyderabad

Problem Set: Functional Series

Teaching Learning Center IIT Hyderabad

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1) Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{4^n n^p}$ w.r.to both x and an arbitrary constant p.
2) Expand in Fourier series the function $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in the interval $[-\pi, \pi]$ and hence show that $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{1}{n^2}$.
3) Show using Taylor's series that $e^{i\theta} = \sin\theta + i\ \cos\theta$.
4) Give examples of the following, with a brief justification.
   (a) A function $f(x)$ and a point x_0 such that $\frac{df}{dx}(x_0) = 0$ but x_0 is not an extreme point.
   (b) A function $f(x)$ and a point x_0 such that x_0 is an extreme point but $\frac{df}{dx}(x_0)$ does not exist.
   (c) A function $f(x)$ and a point x_0 such that $\frac{df}{dx}$ is continuous at x_0 but not differentiable at x_0.
   (d) A power series such that it is convergent only at $x = 1$.
   (e) A function $f(x)$ that is nowhere piecewise monotonic.
   (f) A sequence of functions $\{f_n\}$ defined everywhere (i.e., for all $x \in \mathbb{R}$) but whose functional series $\sum_{n=1}^{\infty} f(x)$ does **not** converge anywhere (i.e., for any $x \in \mathbb{R}$).
   (g) A sequence of functions $\{f_n\}$ to show that the converse of the following statement is **not true**: *"If a sequence of continuous functions $\{f_n\}$ converges uniformly to f, then f is continuous."*
5) Determine the exact intervals of convergence for the following:

   (i) $\sum n^2 x^2$ (vi) $\sum \sqrt{n x^n}$

   (ii) $\sum \frac{2^n}{x^2} x^n$ (vii) $\sum \frac{3^n}{n4^n} x^n$

   (iii) $\sum \frac{x^n}{n^n}$ (viii) $\sum \frac{n^3}{3^n} x^n$

   (iv) $\sum \frac{1}{(n+1)^2 2^n} x^n$ (ix) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

   (v) $\sum \frac{(-1)^n}{n^2 4^n} x^n$ (x) $\sum x^{n!}$
6) Consider a power series $\sum a_n x^n$ with radius of convergence $\mathbb{R}$.
   (a) Prove that if all the coefficients a_n are integers and if infinitely many of them are non-zeros, then $\mathbb{R} \leqslant 1$.
   (b) If $|a_n|^{\frac{1}{n}} \to l$,then $\mathbb{R} = \begin{cases} 0 & l = \infty \\ \infty & l = 0 \\ \frac{1}{l} & 0 < l < \infty \end{cases}$
   (c) If $a_n \neq 0$ for all large n and $\dfrac{|a_{n+1}|}{|a_n|} \to l$,the

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conclusion of (b) above still holds.

(d) Verify the above with the following series whose co-efficients are given as:

(i) $a_n = \dfrac{n^3}{3^n}$

(ii) $a_n = \dfrac{2^n}{n!}$

(iii) $a_{2n-1} = \dfrac{1}{4^n}; a_{2n} = \dfrac{1}{9^n}$

7) Consider a power series $\sum a_n x^n$ with a finite radius of convergence $\mathbb{R}$. Prove that if all the coefficients $a_n \geqslant 0$ for all n and if the series converges at $\mathbb{R}$, then the series also converges at $-\mathbf{R}$.

8) For each $n \in \mathbb{N}$, let $f_n = (\cos x)^n$. Show that

(a) each f_n is continuous.

(b) $\lim f_n(x) = 0$ unless x is a multiple of π.

(c) $\lim f_n(x) = 1$ if x is an even multiple of π.

(d) $\lim f_n(x)$ doesnot exist if x is an odd multiple of π.

9) For each $n \in \mathbb{N}$, let $f_n = \frac{1}{n}\sin x$. Show that

(a) each f_n is differentiable.

(b) $\lim f_n(x) = 0$ for all $x \in \mathbb{R}$.

(c) $\lim f_n'(x)$ need not exist (for instance at $x = \pi$).

10) For each $n \in \mathbb{N}$, let $f_n(x) = nx^n$ for $x \in [0,1]$. Show that

(a) $\lim f_n(x) = 0$ for all $x \in [0,1]$.

(b) $\lim\limits_{n\to\infty} \int\limits_0^1 f_n(x)dx = 1$.

11) For each $n \in \mathbb{N}$, let $f_n(x) = \left(x - \dfrac{1}{n}\right)^2$ for $x \in [0,1]$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Does (f_n) converge pointwise on $[0,1]$?

(c) Does it also converge uniformly?

12) Obtain the Taylor series of the following functions about the indicated point a:

(i) $\tan x; a = \frac{\pi}{4}$

(ii) $e^{\sin x}; a = 0$

(iii) $\ln(\cos x) a = 0$

(iv) $\cos^2 x; a = 0$

(v) $\cos^2 x; a = \frac{\pi}{4}$

(vi) $\dfrac{1}{x^3}; a = 7$

(vii) $\dfrac{a}{1+x^4}; a = 3$

(viii) $\tan^{-1} x; a = 0$

13) Suppose that f_n is differentiable on an interval I centered at $x = a$ and that

$$g(x) = b_0 + b_1(x-a) + ... + b_n(x-a)^n,$$

is a polynomial of degree n with constant coefficients $b_0, b_1, ..., b_n$.Let $E(x) = f(x) - g(x)$. Show that if

(a) $E(a) = 0$ (i.e., the approximation error is zero at $x = a$).

(b) $\lim\limits_{x\to a} \dfrac{E(x)}{(x-a)^n} = 0$ (i.e.,the error is negligible when compared to $(x-a)^n$)

then $b_k = \dfrac{f'(a)}{k!}, k = 0, ..., n$. Thus the Taylor's polynomial is the only polynomial of degree less than or equal to n whose error is zero at $x = a$ and negligible when compared to $(x-a)^n$.

14) Find the Fourier Series of the following functions:

(i) $f_x = x^3; -\pi \leqslant x \leqslant \pi$

(ii) $f_x = x + |x|; x \in [-\pi, \pi]$

(iii) $f_x = \begin{cases} 1; -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ -1; \frac{\pi}{2} < x \leqslant \frac{3\pi}{2} \end{cases}$

(iv) $f_x = \begin{cases} 1; -1 \leqslant x \leqslant 0 \\ -1; 0 < x \leqslant 1 \end{cases}$

(v) $f_x = \begin{cases} x; -2 \leqslant x < 0 \\ \pi - x; 0 < x \leqslant 2 \end{cases}$

(vi) $f_x = x^3; -2 \leqslant x \leqslant 2$

(vii) $f_x = x + |x|; \frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

(viii) $f_x = \begin{cases} x; -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ \pi - x; x \in [-\frac{\pi}{2}, \frac{3\pi}{2}] \end{cases}$

15) Expand the following functions such that we obtain (i) only a sine series and (ii) only a cosine series:

(i) $f_x = x; 0 \leqslant x \leqslant 2\pi$

(ii) $f_x = \pi - x; 0 \leqslant x \leqslant \pi$

(iii) $f_x = \sin^2 x; 0 \leqslant x \leqslant \pi$

(iv) $f_x = e^x; 0 \leqslant x \leqslant L$

(v) $f_x = x^2; 0 \leqslant x \leqslant L$

(vi) $f_x = 4 - x^2; 0 \leqslant x \leqslant L$

16) Find the Fourier Series of $f_x = \dfrac{(\pi - x)^2}{4}$ on $0 \leqslant x \leqslant 2\pi$ and hence show that

(a) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$

(b) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - ...$

(c) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$

17) Find the Fourier Series of $f_x = \sqrt{1 - \cos x}$ on $(0, 2\pi)$ and hence deduce that

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

18) Let f be a periodic function with period 2π. Let f_n be the trignometric polynomial of order n given as follows

$$f_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

19) Show that if f_n minimizes the integral of the square of the error in approximating f,viz.,

$$\int_{-\pi}^{\pi} [f(x) - f_n(x)]^2 dx,$$

then the coefficients of f_n are given as Fourier coefficients.

20) Say True or False, with a brief justification.

(a) The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is dominated on $0 \leqslant x \leqslant 1$.

(b) A function f is Riemann integral over $[a, b]$ if and only if f is continuous over $[a, b]$.

(c) $\int_0^1 x^m(1-x)^n \; dx = \int_0^1 x^n(1-x)^m \; dx$ for any $m > 0, n > 0$.

(d) $e^{i\theta} = \sin\theta + i \;\cos\theta$.

(e) $\int_{-\infty}^{+\infty} f(x) \; dx = \lim_{b\to\infty} \int_{-b}^{+b} f(x) \; dx$ for any real function f.

21) Give example of

(a) a power series whose radius of convergence is $(-3, 3)$.

(b) a functional series that is pointwise convergent but not uniformly convergent.

(c) a function f and an interval $[a, b]$ such that f is not Riemann integral over $[a, b]$.

(d) an infinitely differentiable function whose Taylor's series does not converge to it.

(e) two functions $\phi(x)$ and $\psi(x)$ and an interval $[a, b]$ such that

$\int_a^b \phi(x) \cdot \psi(x) dx = 0$.

22) Find the radius of convergence of the power series $\sum\limits_{n=2}^{\infty} \frac{x^n}{n \ln n}$.

23) Discuss the Maclaurin's series expansion of $f(x) = (1+x)^m$ for any $m \in \mathbb{R}$ and hence find a series expansion of $\sin^{-1} x$.

24) Expand $\frac{1}{1+x^2}$ in powers of x and hence find a power series expansion of $\tan^{-1} x$.

25) Comment on the convergence of the series $\sum a_n$ where $a_n = \begin{cases} \frac{n}{2^n}, & if\ n\ is\ odd \\ \frac{1}{2^n}, & if\ n\ is\ even \end{cases}$.

26) Discuss the Gamma function as an improper integral with respect to its convergence and show that $\Gamma(n+1) = n!$.

27) Expand in Fourier series the function $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in the interval $[-\pi, \pi]$ and hence show that $\frac{\pi^2}{12} = \sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$.

28) Comment on the convergence of the following integrals:

(a) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$ (b) $\int_0^\infty x \sin x\, dx$.

29) Consider the function $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & if\ (x,y) \neq (0,0) \\ 0, & if\ (x,y) = (0,0) \end{cases}$.

- Show that $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$ where $(x,y) \neq (0,0)$.
- Is $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0)$? Substantiate.

30) Evaluate $\int_a^b e^x\, dx$ by calculating it as the limit of Riemann sum.

31) Find the *total* area of a figure bounded by $y = x$, $y = 2x$ and the curve $y = x^3$.

32) Consider the cycloid $x = r(t - \sin t)$; $y = r(1 - \cos t)$.

(a) Find the arc length of one arch $(0 \leqslant t \leqslant 2\pi)$.

(b) Find the surface area of the solid generated by rotating this arch about the x-axis.