Problem Set: Multivariable Calculus

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I. Limits and Partial Derivatives

1) Find the domain and range of the following functions:

(i) $f(x,y) = e^x + e^y$
(ii) $f(x,y) = \frac{x}{y}$
(iii) $f(x,y) = \cos^{-1}(x-y)$
(iv) $f(x,y) = \sqrt{\frac{(x-y)}{(x+y}}$
(v) $f(x,y) = \frac{y}{|x|}$
(vi) $f(x,y) = \frac{x}{2y} + \frac{y}{2x}$
(vii) $f(x,y) = \frac{1}{(x^2-y^2)^{\frac{3}{2}}}$
(viii) $f(x,y,z) = \sqrt{-x^2-y^2-z^2}$
(ix) $f(x,y,z) = \tan^{-1}\left(\frac{x+z}{y}\right)$
(x) $f(x,y,z) = \ln(1+x^2-y^2+z)$

2) Verify the following limits using ϵ - δ definition:

(i) $\lim\limits_{(x,y)\to(1,2)} (3x+y) = 5$
(ii) $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0$

3) Show that the following limits do not exist:

(i) $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x-y}$
(ii) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3}{x^4+y^4}$
(iii) $\lim\limits_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2}$
(iv) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^3+y^3+z^3}$

4) Show that the following limits exist and calculate them:

(i) $\lim\limits_{(x,y)\to(0,0)} \frac{3xy}{\sqrt{x^2+y^2}}$
(ii) $\lim\limits_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^4+y^2}$
(iii) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$
(iv) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$

5) Find the indicated values at the given points:

a) $f(x,y) = \sin(x+y);\; f_x(\pi/6, \pi/3)$
b) $f(x,y) = \ln(x^2+y^4);\; f_y(3,1)$
c) $f(x,y) = e^{\sqrt{x^2+y}};\; f_y(0,4)$
d) $f(x,y) = \frac{x^2-y^2}{x^2+y^2};\; f_y(2,-3)$
e) $f(x,y,z) = \sin(2xy^4z);\; f_{xz}$
f) $f(x,y,z) = e^{xy}\sin z;\; f_{xy}$
g) $f(x,y,z) = \cos(x+2y+3z);\; f_{yz}$
h) $f(x,y) = \ln(3x-2y);\; f_{yxy}$

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II. Continuity, Differentiability and Approximations

1) Find the maximum region over which the following functions are continuous:

(i) $f(x,y) = e^{xy+2}$
(ii) $f(x,y) = \frac{x^3+4xy^6-7x^4}{x^3-y^3}$
(iii) $f(x,y) = \tan^{-1}(x-y)$
(iv) $f(x,y) = \sqrt{x-y}$
(v) $f(x,y,z) = y\ln(xz)$
(vi) $f(x,y,z) = \frac{1}{\sqrt{1-x^2-y^2-z^2}}$

2) Find a function $g(x)$ / a number c such that the following functions are continuous:

(a) $f(x,y) = \begin{cases} \frac{x^2-y^2}{x-y}, & x \neq y \\ g(x), & x = y \end{cases}.$

(b) $f(x,y) = \begin{cases} \frac{3xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ c, & (x,y) = (0,0) \end{cases}.$

(c) $f(x,y) = \begin{cases} \frac{xy}{|x|+|y|}, & (x,y) \neq (0,0) \\ c, & (x,y) = (0,0) \end{cases}.$

3) Show by definition that the following are differentiable:

(i) $z = x^2 + y^2$
(ii) $z = x^2y^2$
(iii) z is any polynomial in x, y

4) Calculate the gradient ∇f of the following functions:

(i) $f(x,y) = (x+y)^2$
(ii) $f(x,y) = \frac{x-y}{x+y}$
(iii) $f(x,y) = \sqrt{x^2+y^3}$
(iv) $f(x,y) = \frac{e^{x^2}-e^{-y^2}}{3y}$
(v) $f(x,y,z) = x\sin y \ln z$
(vi) $f(x,y,z) = \frac{x-z}{\sqrt{1-y^2+x^2}}$
(vii) $f(x,y,z) = x\cosh y - y\ln z$
(viii) $f(x,y,z) = (y-z)e^{\sqrt{1-y^2+x^2}}$

5) Let f and g be differentiable functions of two variables.

(a) Show that $\nabla(f+g) = \nabla f + \nabla g$.
(b) Show that fg is differentiable and that $\nabla(fg) = f\nabla g + g\nabla f$.

(c) Show that $\nabla(f) = \bar{0}$ if and only if f is a constant.
(d) Show that if $\nabla f = \nabla g$ then there is a constant c such that $f(x, y) = g(x, y) = c$.
(e) What is the most general function f such that $\nabla f(\bar{x}) = \bar{x}$ for every $\bar{x} \in \mathbb{R}^2$.

6) Use the total differential to estimate the given numbers:

(i) $\frac{3.01}{5.99}$ (ii) $\sqrt{\frac{5.02-3.96}{5.02+3.96}}$
(iii) $\sin\left(\frac{11\pi}{24}\right)\cos\left(\frac{13\pi}{36}\right)$

7) When 3 resistors r_1, r_2, r_3 are connected in parallel, the total resistance $R = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$. If $r_1 = 6 \pm 0.1$, $r_2 = 8 \pm 0.03$ and $r_3 = 12 \pm 0.15$ ohms, estimate R and find an approximate value for the maximum error in your estimate.
8) How much wood is contained in the sides of a rectangular box with sides of inside measurements $1m$, $1.2m$ and $1.6m$, if the thickness of the wood making up the sides is $5cm$.
9) The volume of 10 moles of an ideal gasa was calculated to be $500cm^3$ at a temperature of $40°C$.If the maximum error in each measurement n, V and T is $\frac{1}{2}\%$, calculate the approximate pressure of the gas (in nt/cm^2), and find the approximate error in your computation.

III. Gradient and Its Applications

1) Calculate the directional derivatives at the given point in the direction of $\bar{v}$:
   (i) $f(x, y) = xy$ at $(2, 3); \bar{v} = \bar{i} + 3\bar{j}$
   (ii) $f(x, y) = \ln(x + 3y)$ at $(2, 4); \bar{v} = \bar{i} + \bar{j}$
   (iii) $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ at $(2, 2); \bar{v} = 3\bar{i} - 2\bar{j}$
   (iv) $f(x, y, z) = x^2y^3 + z\sqrt{x}$ at $(1, -2, 3); \bar{v} = 5\bar{j} + \bar{k}$
   (v) $f(x, y, z) = e^{-(x^2+y^2+z^2)}$ at $(1, 1, 1); \bar{v} = \bar{i} + 3\bar{j} - 5\bar{k}$
   (vi) $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ at $(-1, 2, 3); \bar{v} = \bar{i} - \bar{j} + \bar{k}$
2) The temperature distribution of a ball centered at the origin is given by $T(x, y, z) = \frac{100}{x^2+y^2+z^2+1}$.
   (a) Where is the ball hottest ?
   (b) Find the direction of greatest decrease of heat at the point $(3, -1, 2)$.
   (c) Find the direction of greatest increase in heat. Does this vector point toward the origin ?
3) Determine the nature of the critical points of the given functions:
   (i) $f(x, y) = 7x^2 - 8xy + 3y^2 + 1$
   (ii) $f(x, y) = x^2 + y^3 - 3xy$
   (iii) $f(x, y) = x^3 + 3xy^2 + 3y^2 - 15x + 2$
   (iv) $f(x, y) = \frac{1}{y} - \frac{1}{x} - 4x + y$
   (v) $f(x, y) = xy + \frac{1}{y} + \frac{8}{x}$
   (vi) $f(x, y) = \frac{2}{y} + \frac{1}{x} + 2x + y + 1$
4) Find 3 numbers whose sum is 50 such that the product xy^2z^3 is a maximum.
5) What is the maximum volume of an open-top rectangular box that can be built from α square meters of wood ?
6) Find the dimensions of the rectangular box of maximum volume that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ whose faces are parallel to the coordinate planes.
7) A company uses 2 types of raw materials, I and II, for its product. If it uses x units of I and y units of II it can produce U units of the finished item where $U(x, y) = 8xy + 32x + 40y - 4x^2 - 6y^2$. Each unit of I costs Rs. 10 and each of unit of II costs Rs. 4. Each unit of the product can be sold for Rs. 40. How can the company maximize its profits?
8) Solve the following using Lagrange multipliers:
   (a) Minimum distance from the point $(3, 0, 1)$ to the plane $2x - y + 4z = 5$.
   (b) Find the maximum and minimum values of xyz if (x, y, z) is on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
   (c) Maximize the function $x^3 + y^3 + z^3$ for (x, y, z) on the planes $x + y + z = 2$ and $x + y - z = 3$.
   (d) Show that among all triangles having the same perimeter, the equilateral triangle has the greatest area.
   (e) The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that are closest and farthest from the origin.
9) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the indicated points:
   (i) $z^3 - xy + yz + y^3 - 2 = 0$ at $(1, 1, 1)$
   (ii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$ at $(2, 3, 6)$

(iii) $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ at $(1, \ln 2, \ln 3)$.

IV. Exercises

1) Give examples of the following, with a brief justification.
   (a) A function $f(x, y)$ and a point (x_0, y_0) such that (x_0, y_0) is a saddle point.
   (b) A function $f(x, y)$ such that $\Delta f = 9x^2y^2\hat{i} + 6x^3y\hat{j}$.
   (c) A function $f(x, y)$ such that at the point $(3, 1)$ the direction of steepest increase of f is along the vector $\bar{u} = \hat{i} + 6\hat{j}$.
2) Use total differential to estimate $(1.95)^4(3.04)^3(0.97)^5$.
3) Show by definition that $f(x, y) = xy^2$ is differentiable.
4) Determine the nature of the critical points of $f(x, y) = 9x^2 + 4y^2 - 12xy + 7$.
5) A firm has Rs. $250,000$ to spend on labour and raw materials. The output of the firm is αxy where α is a constant and x,y are the quantity of labour and raw materials consumed. If the unit price of hiring labour is Rs. 5000 and the unit price of raw materials is 2500, find the ratio of x to y that maximizes output.
6) Find the point at which $w = xyz$ attains its maximum w.r.to the constraints $x + y + z = 30$ and $x + y = z$.
7) Show that every line normal to the surface of a sphere passes through its center.
8) Give examples of functions with the following properties:
   (a) A function $f(x, y)$ whose domain is $\{(x, y) \in \mathbb{R}^2 | x \neq 0\}$ and whose range is $(\frac{-\pi}{2}, \frac{\pi}{2})$.
   (b) A function $f(x, y)$ such that $\frac{df}{dx}$ exists at $(0, 0)$ but f is not continuous at $(0, 0)$.
   (c) A function $f(x, y)$ such that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.
   (d) A function $f(x, y)$ such that $f(0, 0)$ is defined but $\nabla f(0, 0)$ is not.
9) Find the domain, range and level curves of $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$.
10) Show that $\lim\limits_{(x,y)\to(0,0)} \frac{y^2-2x}{y^2+2x}$ does not exist.
11) Find the unit vector along which $4x^2 + 9y^2$ increases most rapidly at the point $(2, 1)$.
12) Find a number such that the following function is continuous at the origin:
$$f_(x, y) = \begin{cases} \frac{-2xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ c, & (x, y) = (0, 0) \end{cases}.$$
13) Examine the function $y = 2\sin x + \cos 2x$ for maxima and minima.
14) Let $z = f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$. Also writing $z = g(r, \theta) = f(r\cos\theta, r\sin\theta)$, show that
$$\left(\frac{\delta g}{\delta r}\right)^2 + \frac{1}{r^2}\left(\frac{\delta g}{\delta \theta}\right)^2 = \left(\frac{\delta f}{\delta x}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2.$$
15) Determine the nature of the critical points of $f(x, y) = 2x^3 - 24xy + 16y^3$.
16) How much wood is contained in the sides of a rectangular box with sides of inside measurements $1.5m, 1.3m\ and\ 2m$, if the thickness of the wood making up the sides is $3cm$?
17) A silo is in the shape of a cylinder topped with a cone (much like a circus tent). If the radius of each is $6m$, and the total surface area is $200m^2$ (excluding the base), what are the heights of the cylinder and the cone that maximize the volume enclosed by the silo?