Sampling and Reconstruction

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Contents

***Abstract*—This manual provides theoretical insights into analog to digital (ADC) and digital to analog (DAC) conversion.**

1 Fourier Transform

Problem 1. Let $x(t)$ be a continuous signal with $x(n) = x(nT_s)$ and

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)\delta\left(t - nT_s\right), \tag{1.1}$$

where

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{1.2}$$

$$\delta(t) = 0, t \neq 0 \tag{1.3}$$

Show that

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(t)\delta\left(t - nT_s\right), \tag{1.4}$$

Problem 2. The *Fourier transform* of a signal $g(t)$ is defined as

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t}\, dt \tag{2.1}$$

Find the Fourier transform of $\delta(t)$ and show that

$$\delta\left(t - nT_s\right) \overset{\mathcal{F}}{\rightleftharpoons} e^{-\jmath 2\pi n f T_s} \tag{2.2}$$

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Problem 3. Find the Fourier transform of $\hat{x}(t)$.

Problem 4. If

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f), \tag{4.1}$$

the *inverse* Fourier transform is given by

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{\jmath 2\pi f t}\, df \tag{4.2}$$

Show that

$$x(n) = \int_{-\infty}^{\infty} X(f)e^{\jmath 2\pi n f T_s}\, df \tag{4.3}$$

2 The Fourier Series

Problem 5. Let

$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-\jmath 2\pi n f T_s} \tag{5.1}$$

Show that

$$\hat{X}(f) = \hat{X}\left(f + f_s\right), \quad f_s = \frac{1}{T_s} \tag{5.2}$$

Problem 6. Show that

$$x(n) = \frac{1}{f_s}\int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} \hat{X}(f)e^{\jmath 2\pi n f T_s}\, df \tag{6.1}$$

Problem 7. Use your intuition along with (4.3) and (6.1) to obtain

$$\hat{X} = \frac{X(f)}{T_s}, \quad -\frac{f_s}{2} < f < \frac{f_s}{2} \tag{7.1}$$

and

$$\hat{X}(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - nf_s) \tag{7.2}$$

3 Nyquist Criterion

Problem 8. Let

$$X(f) = \begin{cases} 1 - \frac{|f|}{B} & |f| < B \\ 0 & \text{otherwise} \end{cases}, B = \alpha f_s \tag{8.1}$$

Using (7.2), sketch $\hat{X}(f)$ for $\alpha < 2$ after fixing a particular value of B.

Problem 9. Repeat the above exercise for $\alpha < 2$. Comment.

Problem 10. Let

$$H(f) = \begin{cases} T_s & |f| < f_s \\ 0 & \text{otherwise} \end{cases} \tag{10.1}$$

Find the constraint on α that yields

$$X(f) = \hat{X}(f)H(f) \tag{10.2}$$

This is known as Nyquist's criterion.

4 Shannon's Interpolation Formula

Problem 11. *Convolution* of $\hat{x}(t)$ and $h(t)$ is defined as

$$\hat{x}(t) * h(t) \triangleq \int_{-\infty}^{\infty} \hat{x}(\tau)h(t-\tau)\,d\tau \tag{11.1}$$

Show that

$$\hat{x}(t) * h(t) \overset{\mathcal{F}}{\rightleftharpoons} \hat{X}(f)H(f) \tag{11.2}$$

Problem 12. Show that

$$h(t) * \delta(t - nT_s) = h(t - nT_s) \tag{12.1}$$

Problem 13. Find $h(t)$ from $H(f)$ using (4.2) and sketch it.

Problem 14. Show that

$$x(t) = \sum_{n=-\infty}^{\infty} x(n)\operatorname{sinc}(t - nT_s) \tag{14.1}$$

where

$$\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t} \tag{14.2}$$