Continuity

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***Abstract*—This manual discusses problems related to continuity through examples. Python scripts are provided to supplement the theory.**

Definition 1. For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \tag{0.1}$$

In terms of limits, continuity at a point c can be defined as follows:

$$\lim_{x \to c} f(x) = f(c) \tag{0.2}$$

Problem 1. Show that $f(x) = \sqrt{x}$ is continuous by using the $\epsilon - \delta$ definition.

Proof. Using Definition 1,

$$\left| \sqrt{x} - \sqrt{x_0} \right| = \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| < \left| \frac{x - x_0}{\sqrt{x_0}} \right| < \frac{\delta}{\sqrt{x_0}} = \epsilon \tag{1.1}$$

if $|x - x_0| < \delta$. Hence, $f(x) = \sqrt{x}$ is continutious. Fig. 1 verifies this result.

```
#from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0,10,50)
fx = np.sqrt(x)

plt.plot(x,fx)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
# # # #Comment the following line
plt.savefig('../figs/1.eps')
plt.show()
```

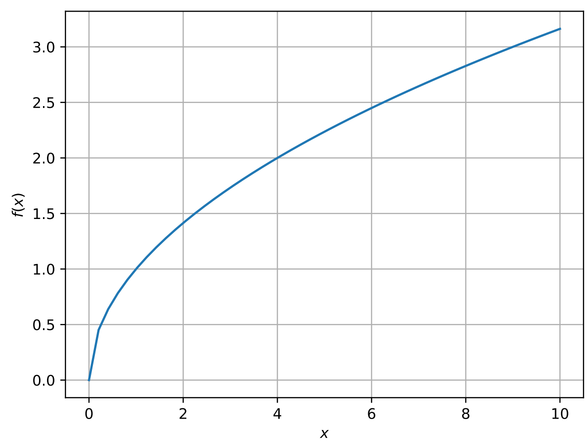

Fig. 1

□

Problem 2. Verify that

$$|\sin x| \le |x| \tag{2.1}$$

Solution: The following code yields Fig. 2.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-np.pi,np.pi,50)
fx = np.abs(np.sin(x))

plt.plot(x,fx,x,np.abs(x),'o')
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$|␣\sin(x)|$')
plt.ylim([0, 1.3])
# # # #Comment the following line
plt.savefig('../figs/2.eps')
plt.show()
```

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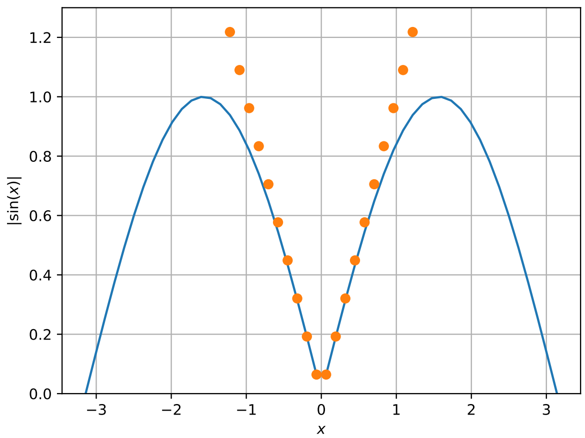

Fig. 2: $|\sin x| \le |x|$

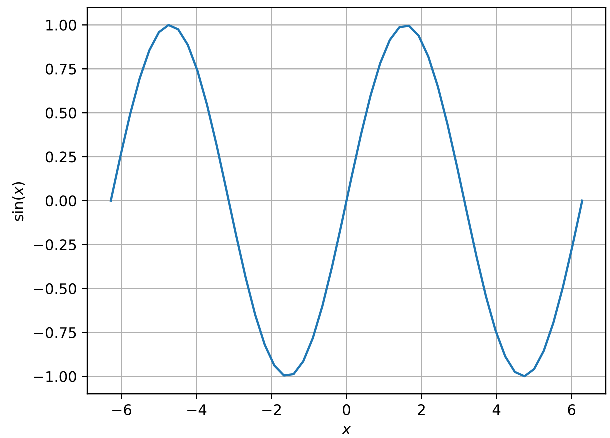

Fig. 3

Problem 3. Show that $f(x) = \sin x$ is continuous.

Proof. Since

$$|\sin x - \sin x_0| = 2\left|\sin\left(\frac{x-x_0}{2}\right)\cos\left(\frac{x+x_0}{2}\right)\right| \tag{3.1}$$

$$\le 2\left|\sin\left(\frac{x-x_0}{2}\right)\right| \tag{3.2}$$

$\because \cos\left(\frac{x+x_0}{2}\right) \le 1$. From Fig. 2,

$$2\left|\sin\left(\frac{x-x_0}{2}\right)\right| < |x - x_0| < \delta = \epsilon \tag{3.3}$$

The following code plots Fig. 3 verifying that $\sin x$ is indeed continuous.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-2*np.pi,2*np.pi,50)
fx = np.sin(x)

plt.plot(x,fx)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$\sin(x)$')
#plt.ylim([0, 1.3])
# # # #Comment the following line
plt.savefig('../figs/2_1.eps')
plt.show()
```

□

Problem 4. Investigate the continuity of the following function at $x = x_0$

$$f(x) = \begin{cases} \sin(\pi x) & 0 < x < 1 \\ \ln(x) & 1 < x < 2 \end{cases}, x_0 = 1 \tag{4.1}$$

Solution: The following code plots $f(x)$ and Fig. 4 indicates that $f(x)$ is indeed continuous at $x_0 = 1$.

```
import numpy as np
import matplotlib.pyplot as plt
n = np.arange(0,2,1e-3)
b=[]
for i in n:
        if(i<=1):
                b.append(np.sin(np.pi*i))
        else:
                b.append(np.log(i))
plt.plot(n,b)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
plt.savefig('../figs/4.eps')
plt.show()
```

Consider the left limit at $x_0 = 1$. So,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin(\pi x) = \sin(\pi) = 0. \tag{4.2}$$

Similarly,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \ln(x) = \ln(1) = 0. \tag{4.3}$$

From Definition 1, $f(x)$ is continuous at $x_0 = 1$.

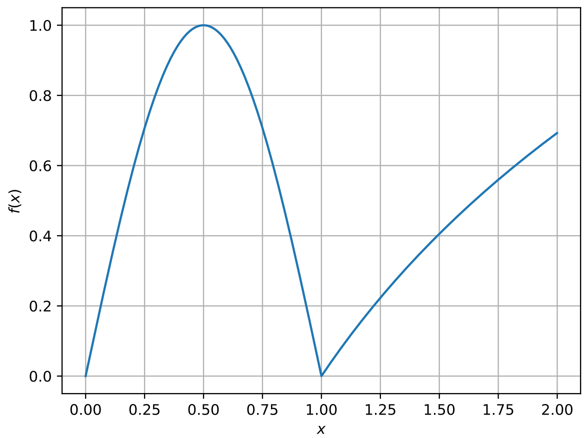

Fig. 4

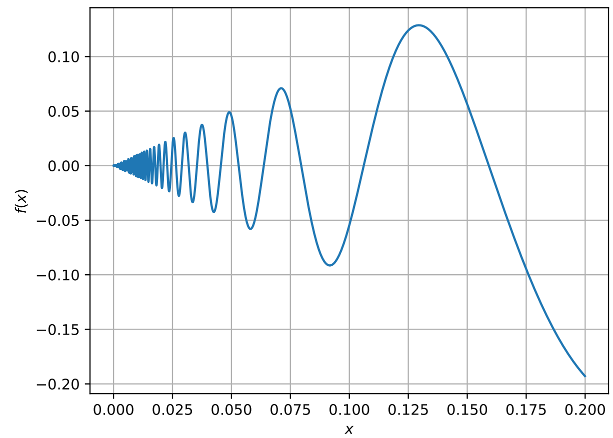

Fig. 5

Proposition 1. If f and g are two continuous functions, then the functions :

- $f+g$
- $f-g$
- fg
- $\frac{f}{g}$

are also continuous wherever they are defined. Note that the similar properties hold for limits at any point a.

Problem 5. Prove that $\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\sin(x)} = 0$.

Solution: The following code plots Fig. 5 indicating that the above result is true.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
x = np.arange(1e-5,0.2,1e-4)
f=((x**2)*np.sin(1/x))/(np.sin(x))
plt.plot(x,f)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
plt.savefig('../figs/5.eps')
plt.show()
```

Rearranging the given function as

$$f(x) = \frac{x\sin\left(\frac{1}{x}\right)}{\frac{\sin(x)}{x}} \tag{5.1}$$

Let

$$g(x) = x\sin\left(\frac{1}{x}\right), h(x) = \frac{\sin(x)}{x} \tag{5.2}$$

By proposition 1,

$$\lim_{x\to 0} f(x) = \frac{\lim_{x\to 0} g(x)}{\lim_{x\to 0} h(x)}. \tag{5.3}$$

$\because \lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} h(x) = 1$, from (5.3), $\lim_{x\to 0} f(x) = \frac{0}{1} = 0$.

Problem 6. Find the domain of continuity of $f(x) = \sqrt{1-x^2}$.

Solution: The following code plots $f(x)$ in the continuity domain. From Fig. 6 it is trivial that $f(x)$ is continuous in its domain.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
x = np.arange(-1,1,1e-3)
f=np.sqrt(1-x**2)
plt.plot(x,f)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
plt.savefig('../figs/6.eps')
plt.show()
```

The function $f(x)$ is continuous for all x for which it is defined. $f(x)$ is defined for x such that

$$1-x^2 \geq 0 \implies |x| \leq 1 \tag{6.1}$$

$\therefore$ the domain of continuity of $f(x)$ is $|x| \leq 1$

Proposition 2. Let I be an interval having the point a as a limit point. Let g, f and h be functions defined

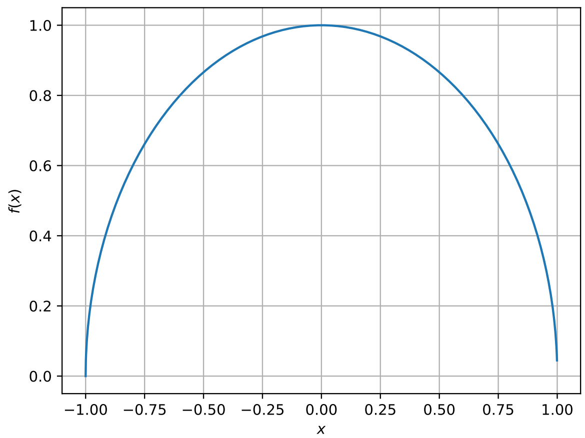

Fig. 6: $f(x) = \sqrt{1 - x^2}$

on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have

$$g(x) \leq f(x) \leq h(x) \tag{6.2}$$

and also if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} f(x) = L$.

Problem 7. Prove that

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 5 & x = 0 \end{cases} \tag{7.1}$$

is discontinuous at $x = 0$ and redefine $f(x)$ such that it is continuous at $x = 0$.

Proof. The following code plots $f(x)$ where $f(0) = 5$ in the upper half and $f(0) = 0$ in the lower half. From upper half of Fig. 7 it is trivial that $f(x)$ is discontinuous at $x = 0$ and if $f(0) = 0$, it is trivial that it is continuous.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
n = np.arange(1e-5,1e-1,1e-3/5)
f=n*np.sin(1/n)
n[0]=0
f[0]=5
plt.figure(1)
plt.subplot(211)
plt.plot(n, f)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
n = np.arange(1e-5,1e-1,1e-3/5)
f=n*np.sin(1/n)
n[0]=0
f[0]=0
plt.subplot(212)
plt.plot(n, f)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
plt.savefig('../figs/7.eps')
plt.show()
```

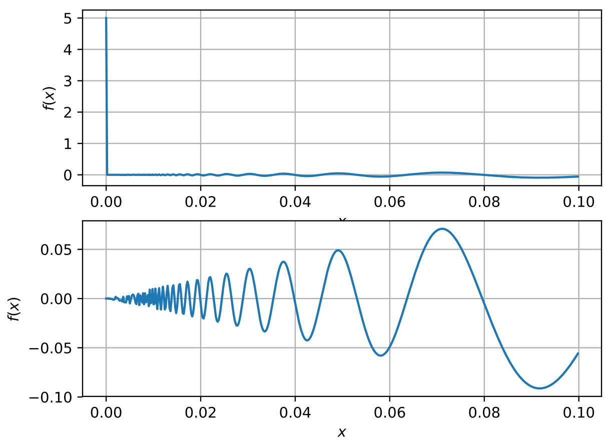

Fig. 7

$\because$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \implies -x \leq x\sin\left(\frac{1}{x}\right) \leq x, x > 0, \tag{7.2}$$

from Proposition 2, as $x \to 0+, x\sin\left(\frac{1}{x}\right) \to 0$. It can be shown that this is true when $x \to 0-$. So $\lim_{x\to 0} f(x) = 0$. According to Definition 1, $\lim_{x\to 0} f(x) = f(0)$. But $f(0) = 5 \neq 0$. $\therefore$ $f(x)$ is discontinuous at $x = 0$. For $f(x)$ to be continuous at $x = 0$ define $f(0) = 0$. □

Problem 8. Prove that all polynomials of finite degree over $\mathbb{R}$ are continuous.

Proof. Consider a function $f(x)$ such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \tag{8.1}$$

Clearly $f(x)$ is a linear combination of some real numbers and the powers of x. So if $g(x)$=x is continuous, then by Proposition 1 we can say that $f(x)$ is continuous. By definition 1 it can be trivially shown that $g(x)$ is continuous. So, $f(x)$ is continuous. $\square$

Problem 9. Give examples of the following:

1) Function which is continuous at finite number of points.
2) Function which is nowhere continuous on $\mathbb{R}$.

Solution:

1)
$$f(x) = \begin{cases} (x-2)(x-3) & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases} \tag{9.1}$$
which is continuous only at $x = 2, x = 3$.

2)
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases} \tag{9.2}$$

Problem 10. Consider two continuous functions $f(x), g(x)$ such that $f(x) = g(x)$ for every rational x. Prove that equality holds for all x in $\mathbb{R}$

Proof. Consider the function
$$h(x) = f(x) - g(x), x \in \mathbb{R} \tag{10.1}$$
By proposition 1, $h(x)$ is continuous at all real x. Consider an arbitrary rational number c and its δ neighborhood. Clearly $h(x)$ is equal to zero at all rational x. $\because h(x)$ is continuous $\lim_{x\to c} h(x) = h(c) = 0$. This implies that $h(x)$ must be zero even in the neighborhood of c which consists of infinite number of irrationals. $\therefore h(x) = 0$ for all $x \in \mathbb{R}$. $h(x) = 0 \implies f(x) = g(x), x \in \mathbb{R}$. $\square$

Proposition 3. Consider a function $f : I \to \mathbb{R}$ where $I = [a, b]$. If $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then $\ni c \in$ I such that $f(c) = 0$.

Problem 11. Show that every polynomial of odd degree with real coefficients has atleast one real root.

Proof. Consider $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$, n is odd. Let $g(x) = \frac{f(x)}{x^n} = \frac{a_0}{x^n} + \frac{a_1x}{x^n} + \frac{a_2x^2}{x^n} + \ldots\ldots + a_n$. Consider the interval $[a, b]$ where $a \ll 0, b \gg 0$ $g(a) \to 1$ as $a \to -\infty$. Similarly $g(b) \to 1$ as $b \to \infty$. $\because n$ is odd, for negative values of x, $x^n < 0$ and for positive values of x it is $x^n > 0$. But $g(x) = \frac{f(x)}{x^n} = 1 > 0$ in both cases. So $f(x) < 0$ at $x = a$ and $f(x) > 0$ at $x = b$. $\therefore$ by proposition 3, $f(x)$ has atleast one root in the interval I. Hence, proved. $\square$

Proposition 4. Consider a function $f : I \to \mathbb{R}$ where $I = [a, b]$. If $f(a) < k < f(b), k \in \mathbb{R}$, then $\ni c \in I$ such that $f(c) = k$.

Problem 12. Let I=[0,1] and $f(0) = f(1)$ where f is continuous on I. Prove that $\ni c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$

Proof. Consider $h(x) = f(x) - f(x + \frac{1}{2}), x \in [0, \frac{1}{2}]$.
$h(0) = f(0) - f(\frac{1}{2})$
$h(\frac{1}{2}) = f(\frac{1}{2}) - f(1) = f(\frac{1}{2}) - f(0) (\because f(0) = f(1))$
$\implies h(\frac{1}{2}) = -(f(0) - f(\frac{1}{2}) = -h(0)$ So $h(0) < 0 < h(1/2)$. By proposition 4, $\ni c \in [0, \frac{1}{2}]$ such that $h(c) = 0$. $\implies f(c) - f(c + \frac{1}{2}) = 0 \implies f(c) = f(c + \frac{1}{2})$. Hence, proved. $\square$

Proposition 5. Every bounded sequence has a convergent sub-sequence.

Problem 13. Let $f : I \to \mathbb{R}$ be a continuous function on I where $I = [a, b]$. Prove that $f(x)$ is bounded.

Proof. Let f be unbounded on I. So, $\ni$ a sequence $x_n \in I$ such that $|f(x_n)| > n$. Because $a \leq x \leq b$, the sequence x_n is bounded. By proposition 5, $\ni$ a sub-sequence x_{n_k} which converges to a value $L \in I$. $\because f$ is continuous, $lim_{k\to\infty} f(x_{n_k}) = f(L)$. This implies that the sequence $f(x_{n_k})$ is convergent $\implies f(x_{n_k})$ is bounded. But this is a contradiction to our assumption that $f(x)$ is unbounded. $\therefore$ every continuous function is bounded on a closed interval. $\square$

Problem 14. Prove that a continuous function attains its bounds on a closed interval I.

Proof. Let M be the least upper bound of the continuous function $f(x)$. We have to prove that $f(x)$ takes the value M for some $c \in I$. $\because M$ is the least upper bound of $f(x)$, the number $M - \frac{1}{n}$ is not the least upper bound of $f(x)$.
$\implies \ni$ a sequence $c_n \in I$ (where $\lim_{n\to\infty} c_n = c$) such that $M - \frac{1}{n} \leq f(c_n) \leq M$.
$\implies \lim_{n\to\infty} f(c_n) = f(c) = M$ (by proposition 2. The above result clearly states that $f(c) = M$ for some $c \in I$. In the similar way, we can prove that it attains the lower bound for some $d \in I$. $\square$