Series

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***Abstract*—This manual covers the convergence/divergence of series through examples. Python scripts are provided for understanding the propagation of the series.**

Problem 1. Sketch the series whose nth term is

$$u_n = \left(\frac{1}{\sqrt{2}}\right)^n \tag{0.1}$$

Solution:

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

n = np.arange(0,50)
s_n=0
b=[]

for i in n:
        T_n = (1/np.sqrt(2))**i
        s_n=s_n+T_n
        b.append(s_n)

plt.stem(n,b)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$S_n$')
###Comment the following line
plt.savefig('../figs/1.eps')
plt.show()
```

Proposition 1. (*Root test*) If

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} < 1 \tag{0.2}$$

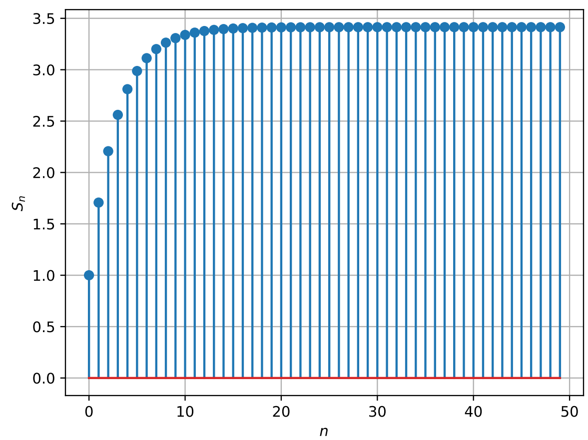

Fig. 1

then the series defined by

$$S_n = \sum_{k=1}^{n} u_k \tag{0.3}$$

converges.

Problem 2. Show that the series in Problem 1 converges using the root test.

Proposition 2. (*Ratio test*)If

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1, \tag{0.4}$$

then S_n converges.

Problem 3. Show that the series in Problem 1 converges using the ratio test.

Problem 4. Graphically examine the series

$$u_n = \sqrt{n+1} - \sqrt{n} \tag{0.5}$$

Solution: From Fig. 4, it can be seen that the series diverges

```
from __future__ import division
```

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```
import numpy as np
import matplotlib.pyplot as plt

n = np.arange(1,100)
s_n=0
b=[1]

for i in range(1,99):
        T_n = np.sqrt(i+1)-np.sqrt
            (i) #this function can
            be changed
        s_n=s_n+T_n
        b.append(s_n)

plt.stem(n,b)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$S_n$')
# # # #Comment the following line
#plt.savefig('../figs/3.eps')
plt.show()
```

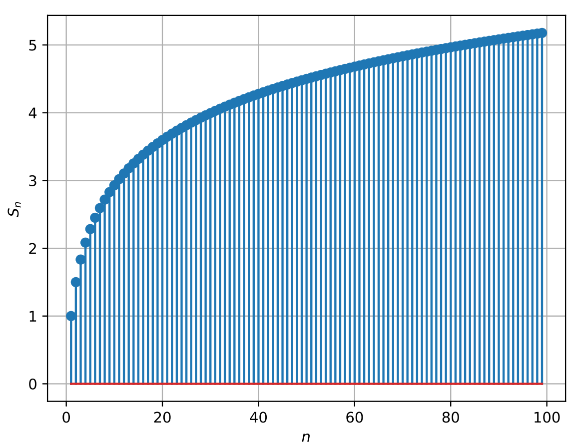

Fig. 4

Problem 5. Show that the series defined by

$$u_n = \sqrt{n+1} - \sqrt{n} \tag{0.6}$$

diverges.

Proof. Since

$$S_n = \sum_{i=1}^{n} u_k = \sqrt{n+1} - 1, \tag{0.7}$$

which is monotonically increasing as well as unbounded, S_n diverges. □

Proposition 3. Let the nth terms of two series be a_n and b_n. If $a_n < b_n$

1) and the b_n series converges, then the a_n series also converges.
2) and the a_n series diverges, then the b_n series diverges.

Problem 6. Sketch the series defined by

$$u_n = \frac{2^n - 1}{3^n} \tag{0.8}$$

Solution: From Fig. 6, it can be seen that the series converges.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

n = np.arange(0,50)
s_n=0
b=[]

for i in n:
        T_n=(2/3)**i -(1/3)**i
        s_n=s_n+T_n
        b.append(s_n)
plt.stem(n,b)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$S_n$')
# # # #Comment the following line
#plt.savefig('../figs/5.eps')
plt.show()
```

Problem 7. Show that S_n in Problem 6 converges.

Solution: Since

$$u_n = \frac{2^n - 1}{3^n} < \frac{2^n}{3^n}, \tag{0.9}$$

using the root test, the $\frac{2^n}{3^n}$ series converges and from Proposition 3, S_n converges.

Proposition 4. If u_n is nonnegative and nonincreasing, then S_n converges if and only if the $2^n u_{2^n}$ series converges.

Problem 8. Graphically examine the series

$$u_n = \frac{1}{n\sqrt{n+1}} \tag{0.10}$$

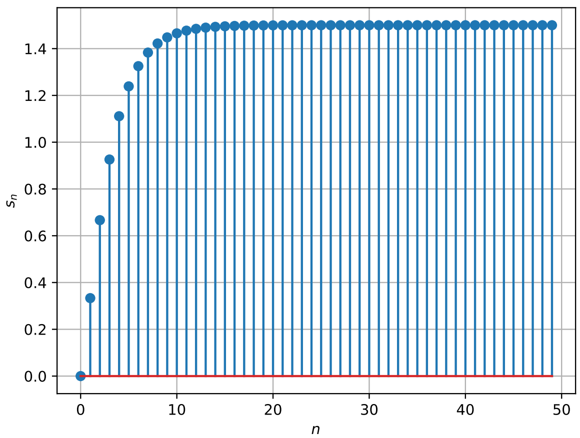

Fig. 6

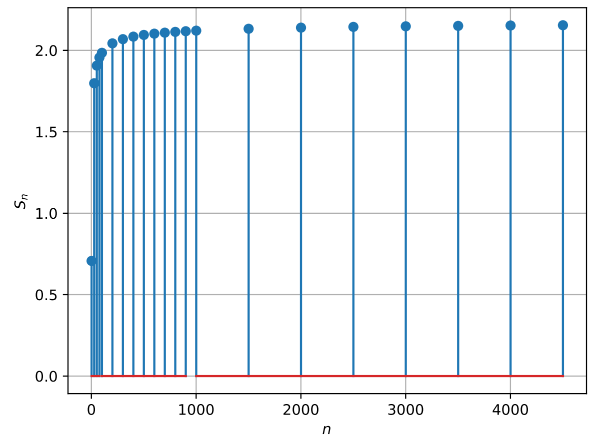

Fig. 8

Solution: From Fig. 8, it can be seen that the series converges.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

n = np.arange(1,10001)
s_n=0
b=[0]
#print(n)
for i in range(1,10000):
        T_n = 1/(i*np.sqrt(i+1)) #
            this function can be
            changed
        s_n=s_n+T_n
        b.append(s_n)

print(s_n)

plt.stem(n[1:100:25],b[1:100:25])
plt.stem(n[100:1000:100],b
   [100:1000:100])
plt.stem(n[1000:5000:500],b
   [1000:5000:500])
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$S_n$')
# # # #Comment the following line
#plt.savefig('../figs/6.eps')
plt.show()
```

Problem 9. Show that

$$u_n = \frac{1}{n\sqrt{n+1}} \tag{0.11}$$

converges.

Solution: It is obvious that

$$u_n = \frac{1}{n\sqrt{n+1}} < \frac{1}{n\sqrt{n}} = f(n), \text{ say} \tag{0.12}$$

Then,

$$2^n f(2^n) = \frac{2^n}{2^{\frac{3n}{2}}} \tag{0.13}$$

$$= \frac{1}{\left(\sqrt{2}\right)^n} \tag{0.14}$$

which yields a convergent series using the root test. From Proposition 3, it is obvious that S_n converges.

Proposition 5. S_n diverges if $\lim_{n\to\infty} u_n \neq 0$.

Problem 10. Graphically examine the series

$$u_n = \frac{n}{n+1} \tag{0.15}$$

Solution: From Fig. 10, it can be seen that the series diverges.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

n = np.arange(1,100)
s_n=0
b=[0]
```

```
for i in range(1,99):
        T_n = i/(i+1) #this
            function can be changed
        s_n=s_n+T_n
        b.append(s_n)

plt.stem(n,b)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$S_n$')
# # # #Comment the following line
plt.savefig('../figs/4.eps')
plt.show()
```

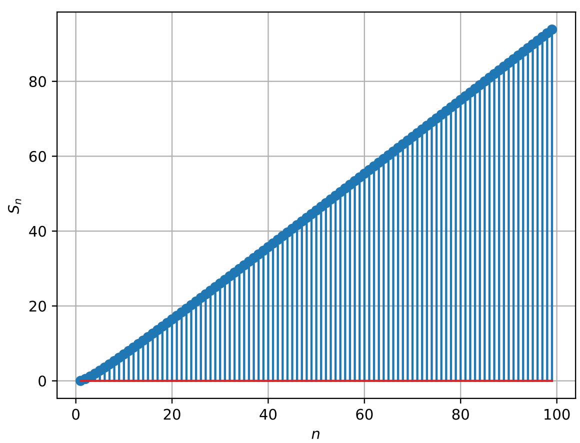

Fig. 10

Problem 11. Show that

$$u_n = \frac{n}{n+1} \tag{0.16}$$

series diverges.

Proof. Trivial using Propostion 5. □

Proposition 6. (*Integral test*) Let $f(n) = u_n$ be positive and monotone decreasing. Then S_n converges if

$$\lim_{t\to\infty} \int_1^t f(t)\, dt < \infty \tag{0.17}$$

If the integral diverges, then S_n also diverges.

Problem 12. Graphically examine the series

$$u_n = \frac{1}{n} \tag{0.18}$$

Solution: From Fig. 12, it can be seen that the series diverges.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

n = np.arange(1,100)
s_n=0
b=[]
for i in range(1,100):
        T_n = 1/i #this function
            can be changed
        s_n=s_n+T_n
        b.append(s_n)

plt.stem(n,b)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$S_n$')
# # # #Comment the following line
#plt.savefig('../figs/2.eps')
plt.show()
```

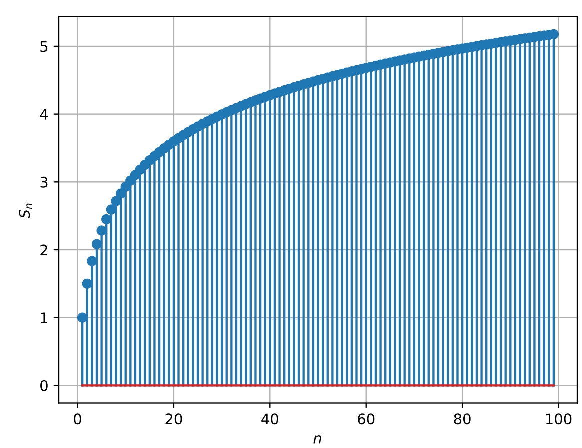

Fig. 12

Problem 13. Show that

$$u_n = \frac{1}{n} \tag{0.19}$$

series diverges.

Proof. Using the integral test,

$$\lim_{t\to\infty} \int_1^t \frac{1}{t}\, dt = \lim_{t\infty} \ln t \tag{0.20}$$

which does not converge. Hence the series diverges. □

Proposition 7. (*Leibniz Test*) If

$$u_n = (-1)^n a_n, \tag{0.21}$$

then S_n converges if

1) a_n is monotonically decreasing
2) $\lim_{n\to\infty} a_n = 0$

Problem 14. Graphically examine the series

$$u_n = \frac{\cos n\pi}{\sqrt{n}} \tag{0.22}$$

Solution: From Fig. 14, it can be seen that the series converges.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

k=np.pi
n = np.arange(1,200)
s_n=0
b=[]
for i in range(1,200):
        T_n = np.cos(k*i)/np.sqrt(
            i) #this function can be
             changed
        s_n=s_n+T_n
        b.append(s_n)
print(s_n)
plt.stem(n,b)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$S_n$')
# # # #Comment the following line
#plt.savefig('../figs/7.eps')
plt.show()
```

Problem 15. Show that

$$u_n = \frac{\cos n\pi}{\sqrt{n}} \tag{0.23}$$

converges.

Proof. $\frac{1}{\sqrt{n}}$ is monotonically decreasing and goes to 0 as $n \to \infty$. Using Proposition 7, S_n converges. □

Definition 1. The series $\sum_{n=0}^{m} u_n$ converges conditionally if $\lim_{m\to\infty} \sum_{n=0}^{m} u_n < \infty$ but $\sum_{n=0}^{\infty} |u_n| = \infty$.

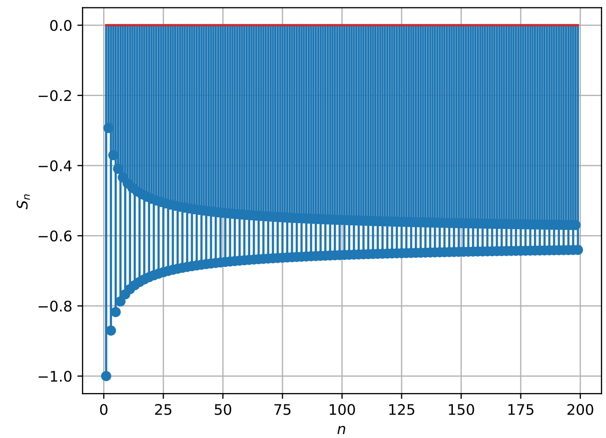

Fig. 14

Problem 16. Comment on the convergence of the series in Problem 14.

Solution: It was shown earlier that the

$$u_n = \frac{\cos n\pi}{\sqrt{n}} \tag{0.24}$$

converges. For absolute convergence, it is necessary that

$$|u_n| = \frac{1}{\sqrt{n}} \tag{0.25}$$

series converge. The following script generates Fig. 16, indicating that the $|u_n|$ series diverges.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

k=np.pi
n = np.arange(1,100)
s_n=0
b=[]
for i in range(1,100):
        T_n = 1/np.sqrt(i) #this
            function can be changed
        s_n=s_n+T_n
        b.append(s_n)
print(s_n)
plt.stem(n,b)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$S_n$')
# # # #Comment the following line
#plt.savefig('../figs/8.eps')
```

```
plt.show()
```

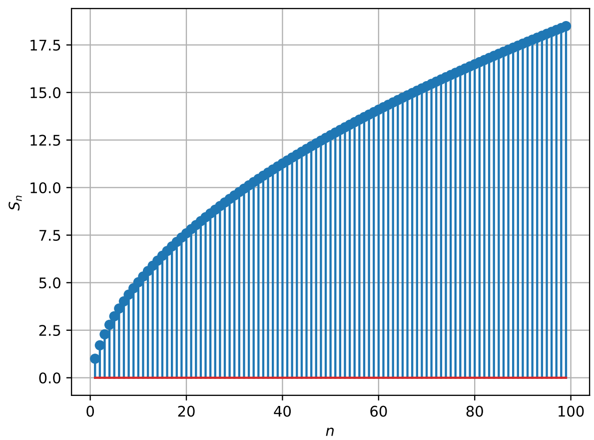

Fig. 16

Since

$$\frac{1}{\sqrt{n}} > \frac{1}{n}, \tag{0.26}$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, from Proposition 3, $\sum_{n=1}^{\infty} |u_n|$ diverges. Hence u_n is conditionally convergent.