Functional Series

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***Abstract*—This manual discusses problems related to functional series through examples. Python scripts are provided to supplement the theory.**

Definition 1. Consider a function $f(x)$ such that its power series expansion at $c \in \mathbb{R}$ is given by:

$$f(x) = \sum_{0}^{\infty} a_n (x-c)^n \tag{0.1}$$

The real number R is said to be the radius of convergence of $f(x)$ if for every x in the interval $(-R+c, R+c)$, $f(x)$ converges. The interval is called the interval of convergence. The radius of convergence can be found by using the condition for convergence in either root test or ratio test(because the power series itself is a series in n). The following is a proof using ratio test.

Proof.

$$f(x) = \sum_{0}^{\infty} a_n (x-c)^n \tag{0.2}$$

Applying ratio test condition for the convergence of $f(x)$

$$\implies \lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n (x-c)^n} \right| < 1 \tag{0.3}$$

Let

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R(>0) \implies \frac{|x-c|}{R} < 1 \tag{0.4}$$

$\implies$ $|x-c| < R$ The above inequality gives the interval $[-R+c, R-c]$ in which $f(x)$ converges depending on R value. If $R=0$ then $f(x)$ converges nowhere on $\mathbb{R}$ and if $R=\infty$, then $f(x)$ converges everywhere on $\mathbb{R}$. Similarly conclusion can be achieved using root test. □

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The following is an example for finding the interval and radius of convergence.

Problem 1. Find the interval of convergence of the power series

$$f(x) = \sum_{0}^{\infty} \frac{n^3}{3^n} x^n. \tag{1.1}$$

Solution:

Using Definition 1,

$$a_n = \frac{n^3}{3^n} \implies R = \lim_{n\to\infty} \frac{n^3}{3^n} \frac{3^{n+1}}{(n+1)^3} = \lim_{n\to\infty} \frac{3n^3}{(n+1)^3} = 3 \tag{1.2}$$

$\therefore R = 3$. So, the radius of convergence is 3 and the interval being $(-3, 3)$. The following graph plots the power series $\sum_0^\infty \frac{n^3}{3^n} x^n$. Clearly for $|x| \geq 3$, the series diverges rapidly whereas it takes values smaller values for x in the interval $(-3, 3)$.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

x = np.arange(-3.0001,3.0001,1e-3)
y1=[]
for x1 in x:
        z=0
        for i in range(1,10):
                z=z+(i**3/3**i)*(
                    x1**i)
        y1.append(z)
plt.plot(x,y1)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
# # # #Comment the following line
plt.savefig('../figs/1.eps')
plt.show()
```

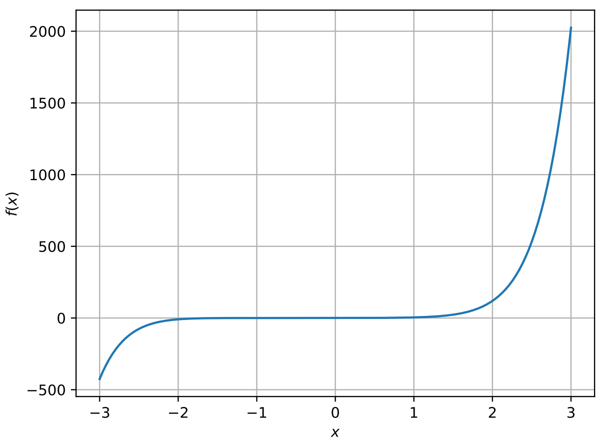


Fig. 1

Definition 2. Let D be a subset of $\mathbb{R}$ and let f_n be a sequence of real valued functions defined on D. Then f_n converges pointwise to f if given any x in D and given any $\epsilon > 0$, there exists a natural number $N = N(x, \epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n > N$. The above definition implies uniform convergence if the inequality holds for every $x \in D$.

Problem 2. For each $n \in \mathbb{N}$, let $f_n(x) = \left(x - \frac{1}{n}\right)^2$ for $x \in [0, 1]$. Find the following:

- Does f_n converge pointwise on [0,1].
- Does it converge uniformly.

Solution:

$$\lim_{n\to\infty} f_n(x) = f(x) = x^2 \tag{2.1}$$

for x in [0,1]. Consider $|f_n(x) - f(x)|$. Since $x \in [0, 1]$, there exists $n \in \mathbb{N}$ such that $x = \frac{1}{n} \implies$

$$|f_n(x) - f(x)| = \left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| \tag{2.2}$$

$\implies$

$$|f_n(x) - f(x)| = \frac{1}{n^2} \tag{2.3}$$

CLearly, $\frac{1}{n^2}$ is greater than ϵ for some values of n. So, $f_n(x)$ does not converge uniformly. The following code proves our first result Fig. 2.

```
import numpy as np
import matplotlib.pyplot as plt

n=1e5
x = np.arange(0,1,1e-3) #for some
    x less than 1 can be changed.
fx=(x-1/n)**2

plt.plot(x,fx)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$f_n$')
# # # #Comment the following line
plt.savefig('../figs/2.eps')
plt.show()
```

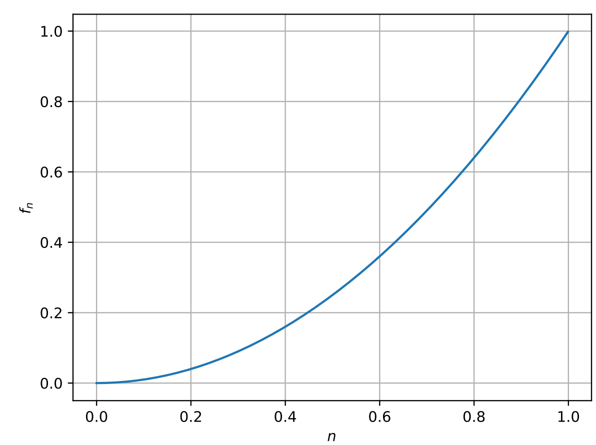


Fig. 2

Definition 3. Taylor Series: The taylor series of a n-time differentiable function $f(x)$ at a point a is given by the following equation: Note: Maclaurian series is nothing but taylor series of $f(x)$ at $x = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} x^n \tag{2.4}$$

Problem 3. Obtain the taylor series of $f(x) = e^x$ at $x = 0$.

Solution: Clearly, $f^n(0) = 1$ for $f(x) = e^x$. So

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{3.1}$$

The following code plots Fig. 3 verifying that the taylor expansion is indeed correct.

```
from __future__ import division
import numpy as np
import math as m
import matplotlib.pyplot as plt
```

```
x = np.linspace(0,1,1000)
y1=[]
x1=np.array(x)
x=np.linspace(0,1,10)
x2=np.array(x)
y=np.exp(x1)
for x in x:
        z=1
        for i in range(1,6):
                z=z+x**i/m.
                   factorial(i)
        y1.append(z)

plt.plot(x1,y)
plt.plot(x2,y1,'o')
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
# # # #Comment the following line
plt.savefig('../figs/3.eps')
plt.show()
```

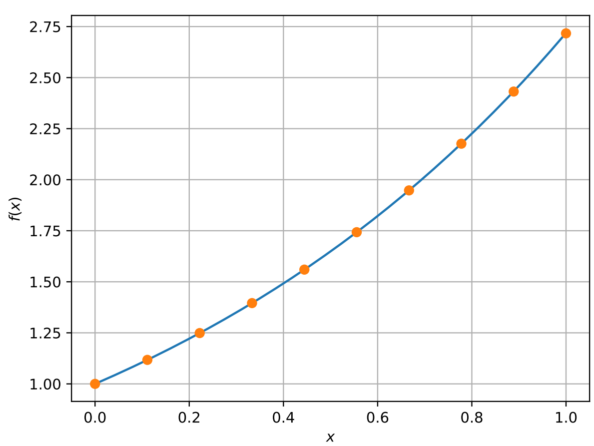


Fig. 3

Definition 4. For a periodic function $f(x)$(with period $2L$), the fourier series is given by the following equation:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right) \quad (3.2)$$

where the coefficients are given by:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \quad (3.3)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nx)\,dx \quad (3.4)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(nx)\,dx \quad (3.5)$$

where $n \geq 1$.

Problem 4. Find the fourier series expansion of $f(x) = \sqrt{1-\cos x}$ on $(0, 2\pi)$ and prove the following result:

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1} \quad (4.1)$$

Solution: Rewriting $f(x)$ interms of $\sin\left(\frac{x}{2}\right) \implies f(x) = \sqrt{2}\sin\left(\frac{x}{2}\right)$. Clearly, $\frac{x}{2} \in (0, \pi)$ where $\sin\left(\frac{x}{2}\right)$ is positive. So, we need to expand $f(x) = \sqrt{2}\sin\left(\frac{x}{2}\right)$. By careful application of 4, we can conclude the following:

$$a_0 = \frac{4\sqrt{2}}{\pi} \quad (4.2)$$

$$a_n = \frac{-4\sqrt{2}}{\pi(4n^2-1)} \quad (4.3)$$

and that $b_n = 0 \implies$

$$\sqrt{1-cosx} = \frac{4\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(4n^2-1)} \quad (4.4)$$

$atx = 0$, the equation becomes

$$\frac{4\sqrt{2}}{\pi} = \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(4n^2-1)} \quad (4.5)$$

$$\implies \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1} \quad (4.6)$$

Hence, proved. Some similar results can also be proved using fourier series.

Problem 5. Suppose

$$f_n(x) = A_0 + \sum_{k=1}^{n}(A_k \cos kx + B_k \sin kx) \quad (5.1)$$

be a trigonometric polynomial and $f(x)$ be a periodic function with period=2π. Show that if $f_n(x)$ minimises the integral $E = \int_{-\pi}^{\pi}(f(x) - f_n(x))^2$, then the coeffecients in $f_n(x)$ are same as the coefficients of the fourier series of $f(x)$.

Proof. Let $a_0, a_1, ..., a_n$ and $b_1, b_2, ..., b_n$ be the fourier series coeffecients of $f(x)$.

Consider the given integral. Expanding it gives $\int_{-\pi}^{\pi}\left(f^2 - 2ff_n + f_n{}^2dx\right)$. It can be trivially shown that the following equations hold true:

$$\int_{-\pi}^{\pi} \cos kx = 0 \tag{5.2}$$

$$\int_{-\pi}^{\pi} \cos^2 kx = \pi \tag{5.3}$$

$$\int_{-\pi}^{\pi} \cos kx \sin kx = 0 \tag{5.4}$$

Same results hold true for $\sin kx$. We can prove the required result by using the above equations. Consider $\int_{-\pi}^{\pi} f_n{}^2dx$. On applying the above equations and simplifying, we end up with the following expression:

$$\int_{-\pi}^{\pi} f_n{}^2dx = \pi\left(2A_0^2 + \sum_{k=1}^{n}\left(A_k^2 + B_k^2\right)\right) \tag{5.5}$$

On similar lines, we can get the following expression for $\int_{-\pi}^{\pi} ff_ndx$

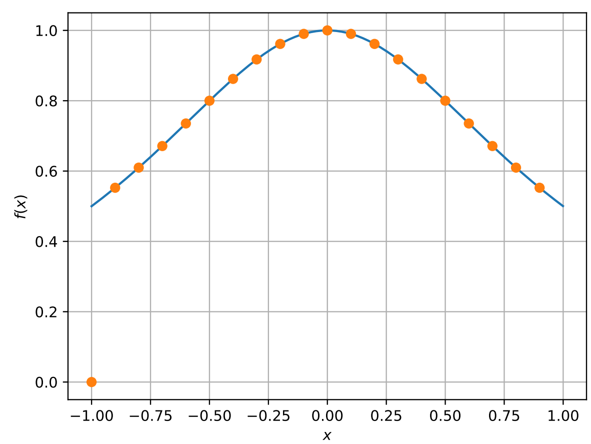
$$\int_{-\pi}^{\pi} ff_ndx = 2\pi\left(2A_0a_0 + \sum_{k=1}^{n}(A_ka_k + B_kb_k)\right) \tag{5.6}$$

On careful observation, one can conclude that the above equation is nothing but the dot product between the vectors $(A_0, A_0, A_1, A_2, ..., B_1, B_2, ...), (a_0, a_0, a_1, a_2, ..., b_1, b_2, ...)$ whose maximum will minimise the error($\because$ it has a negative sign in the expansion). This happens only if all the corresponding components are equal. $\implies$ for minimum error, $A_k = a_k, 0 \le k \le n$. □

Problem 6. Expand $\frac{1}{1+x^2}$ in powers of x and hence find a power series expansion for $\tan^{-1} x$.

Solution: The following code plots $\frac{1}{1+x^2}$ and its power series expansion in $(-1, 1)$ which verifies our answer.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
x = np.arange(-1,1,1e-3)
x1=np.arange(-1,1,1e-1)
f=1/(1+x**2)
f1=[]
for x2 in x1:
        y=0
        for j in range(0,100):
                y=y+((-1)**j)*(x2
                    **(2*j))
        f1.append(y)
plt.plot(x,f)
plt.plot(x1,f1,'o')
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
plt.savefig('../figs/6.eps')
plt.show()
```



$\because$ we are expanding in powers of x, we can safely assume that $|x| < 1$. So, by using GP's infinite summation formula, $\implies$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1-x^2+x^4-x^6+.....(\infty) \tag{6.1}$$

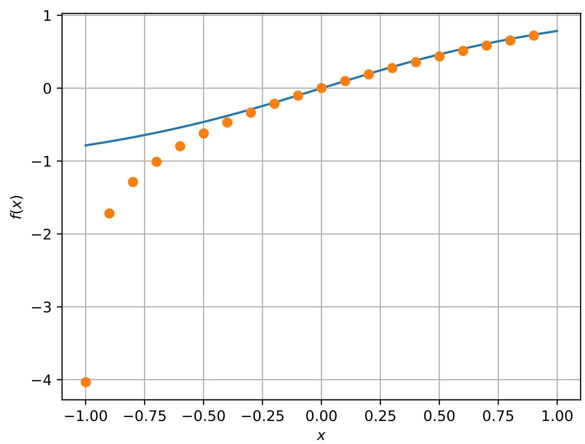
Integrating(indefinite) on both sides($\because$ both the sides of the equation are continuous), $\implies$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \tag{6.2}$$

The above equation is the power series expansion for $\tan^{-1} x$ at $x = 0$. The following figure verifies the above equation.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
x = np.arange(-1,1,1e-3)
x1=np.arange(-0.999,0.9999,1e-1)
f=np.arctan(x)
f1=[]
for x2 in x1:
        y=0
```

```
            for j in range(1,1000):
                    y=y+((-1)**(j+1))
                        *(x2**j/(2*j-1))
            f1.append(y)
plt.plot(x,f)
plt.plot(x1,f1,'o')
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
plt.savefig('../figs/7.eps')
plt.show()
```



Definition 5. Riemann Sum: Suppose $f : [a, b] \to R$ and partition $P = [x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$, then the sum $S = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ is called Riemann Sum over the interval $[a, b]$ whose limit at ∞ gives the definite integral of $f(x)$ over the interval $[a, b]$.

Problem 7. Evaluate $\int_a^b e^x dx$ using Riemann Sum method.

Solution: Let $x_i^* = a + i\Delta x$ where $\Delta x = \frac{b-a}{n}$ where n is the number of partitions. $\implies f(x_i^*) = e^a e^{i\Delta x}$. Doing the summation, from 0 to n gives the following equation:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = e^a \left(\frac{e^{b-a} - 1}{\frac{e^{\frac{b-a}{n}} - 1}{\frac{b-a}{n}}} \right) \tag{7.1}$$

On applying the limit($n \to \infty$), the summation boils down to $e^b - e^a$ which is the required value($\because$ the denominator tends to 1 as $n \to \infty$).

Definition 6. In mathematical analysis, an improper integral is the limit of a definite integral as an endpoint of the interval(s) of integration approaches either a specified real number or ∞ or $-\infty$ or, in some cases, as both endpoints approach limits. Symbolically, it is written as follows:

$$\lim_{b\to\infty} \int_a^b f(x)dx, \lim_{a\to-\infty} \int_a^b f(x)dx \tag{7.2}$$

Similar representation can be given for integrals over finite intervals. If the limit value is finite, then the integral is said to converge to that finite value.

Problem 8. Comment on the convergence of $\int_0^\infty x \sin x$.

Solution: Consider the following integral $\int_0^t x \sin x$ $\implies$

$$\int_0^t x \sin x = -t \cos t + \sin t \tag{8.1}$$

Clearly as $t \to \infty$, the value of the integral diverges to $-\infty$. $\therefore$ the integral diverges.

Proposition 1. The function of the form $\int_0^\infty t^{s-1} e^{-t} dt$, $s > 0$ is called the gamma function. It is denoted by $\Gamma(s)$.

Problem 9. Discuss the convergence of Γ function and prove that $\Gamma(n + 1) = n!$.

Solution: Using 1,

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt = \int_0^1 t^{s-1} e^{-t} dt + \int_1^\infty t^{s-1} e^{-t} dt \tag{9.1}$$

Here $\int_0^1 t^{s-1} e^{-t} dt < \int_0^1 t^{s-1} dt$ which converges to $\frac{1}{s}$. For the second term, we can use the fact that exponential function grows faster than any polynomial. So, $\exists N$ such that for $t \geq N$, $t^{s-1} < e^{\frac{t}{2}}$. Again the integral can be split as

$$\int_1^\infty t^{s-1} e^{-t} dt = \int_1^N t^{s-1} e^{-t} dt + \int_N^\infty t^{s-1} e^{-t} dt \tag{9.2}$$

which is less than

$$\int_1^N t^{s-1} e^{-t} dt + \int_N^\infty e^{-t/2} dt \tag{9.3}$$

Clearly, the above equation takes a finite value. So, $\int_1^\infty t^{s-1} e^{-t} dt < \infty$. $\therefore$ Γ function converges. On applying integration by parts and simplifying we end up with the following equation

$$\Gamma(n) = (n - 1)\Gamma(n - 1) = (n - 1)!\Gamma(1) \tag{9.4}$$

It is trivial that $\Gamma(1) = 1$. So, $\Gamma(n) = (n - 1)!$ which

$\implies \Gamma(n+1) = n!$.

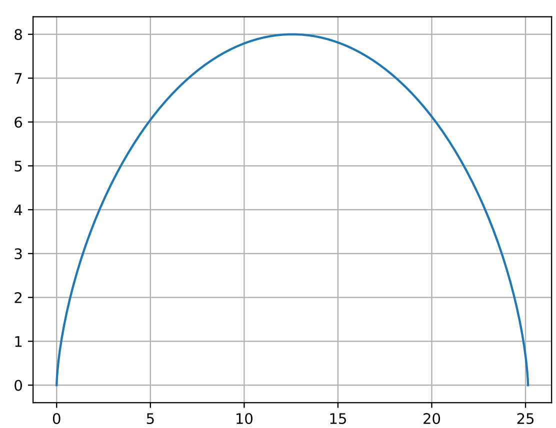
Problem 10. Calculate the following for the cycloid formed by $x = r(t - \sin t), y = r(1 - \cos t)$

- Arc length of one arc$(0 \le t \le 2\pi)$
- Surface area of the solid generated by rotating this arc about the $x-$ axis.

Solution: The following code plots the given curve for $0 \le t \le 2\pi$.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

k=np.pi
t = np.arange(0,2*np.pi,1e-3)
x=4*(t-np.sin(t))
y=4*(1-np.cos(t))
b=[]
plt.plot(x,y)
plt.grid()
# # # #Comment the following line
plt.savefig('../figs/8.eps')
plt.show()
```



We know that $dr = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$. So, total arc length is nothing but $\int_0^{2\pi} dr$. By careful simplification, we get:

$$\frac{dx}{dt} = r(1 - \cos t) \tag{10.1}$$

$$\frac{dy}{dt} = r \sin t \tag{10.2}$$

Substituting the above in the required integral yields

$$\int_0^{2\pi} 2r \sin \frac{t}{2} \tag{10.3}$$

Whose value is $8r$. So, the arc length is $8r$. The surface area is given by the integral $\int_0^{2\pi} 2\pi y dr$. So, the surface area$=\int_0^{2\pi} 2\pi 2r^2(1-\cos t) \sin \frac{t}{2}$. On proper simplification, this integral yields $8\pi r^2$. So, the arc length is $8r$ and the surface area is $8\pi r^2$.

Proposition 2. Let $f_n(x)$ be a sequence of integrable functions on I such that each f_n is non-negative on I and f_n converges(pointwise) to f. Then f_n is said to be dominantly convergent on I if there is some other integrable function on I such that $|f(x)| < g(x)$ almost everywhere on I. The above implies that

$$\lim_{n\to\infty} \int_I f_n(x)dx = \int_I \lim_{n\to\infty} f_n(x)dx \tag{10.4}$$

Similar kind of definition is applicable for series.

Problem 11. Prove or disprove that the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is dominant on $[0, 1]$.

Solution: By 1, it is clear that the given series converges in $[0, 1]$. Also,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} < \sum_{n=1}^{\infty} x^n \tag{11.1}$$

$\because x < 1$,

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \tag{11.2}$$

The above function is integrable on $[0, 1]$ and we also know that the series converges to some function f which will be less than $\frac{x}{1-x}$. $\therefore$ the above function is dominated on $[0, 1]$.