Circuit Analysis

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Contents

***Abstract*—This manual provides a quick introduction to Fourier series and Low Pass Filters (LPF), besides facilitating the use of Python for Signals & Systems.**

1 Filter

1.1 Node Analysis

Problem 1.1. Refer to the circuit in Fig. 1.1. Suppose you are told that C has a resistance given by $\frac{1}{sC}$. Find the ratio $H(s)$ of the output voltage and input voltage using node analysis. The above circuit is known as a low pass filter and $H(s)$ is known as the transfer function.

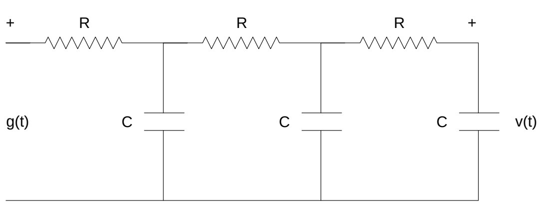

Fig. 1.1: Three stage $R-C$ low pass filter circuit

Solution: The equations at the nodes are given by

$$\frac{V_1 - V_i}{R} + sCV_1 + \frac{V_1 - V_2}{R} = 0 \tag{1.1}$$

$$\frac{V_2 - V_1}{R} + sCV_2 + \frac{V_2 - V_o}{R} = 0 \tag{1.2}$$

$$\frac{V_o - V_2}{R} + sCV_o = 0 \tag{1.3}$$

which can be expressed as

$$\begin{pmatrix} sC + \frac{2}{R} & -\frac{1}{R} & 0 \\ -\frac{1}{R} & sC + \frac{2}{R} & -\frac{1}{R} \\ 0 & -\frac{1}{R} & sC + \frac{1}{R} \end{pmatrix} \begin{pmatrix} \frac{V_1}{V_i} \\ \frac{V_2}{V_i} \\ \frac{V_o}{V_i} \end{pmatrix} = \begin{pmatrix} \frac{1}{R} \\ 0 \\ 0 \end{pmatrix} \tag{1.4}$$

Thus,

$$H(s) = \frac{V_o}{V_i} = \frac{\begin{vmatrix} sC + \frac{2}{R} & -\frac{1}{R} & \frac{1}{R} \\ -\frac{1}{R} & sC + \frac{2}{R} & 0 \\ 0 & -\frac{1}{R} & 0 \end{vmatrix}}{\begin{vmatrix} sC + \frac{2}{R} & -\frac{1}{R} & 0 \\ -\frac{1}{R} & sC + \frac{2}{R} & -\frac{1}{R} \\ 0 & -\frac{1}{R} & sC + \frac{1}{R} \end{vmatrix}} \tag{1.5}$$

$$= \frac{1/R^3}{\left(sC + \frac{1}{R}\right)\left\{\left(sC + \frac{2}{R}\right)^2 - \frac{1}{R^2}\right\} - \frac{1}{R^2}\left(sC + \frac{2}{R}\right)} \tag{1.6}$$

which can be expressed as

$$H(s) = \frac{1}{(sCR + 1)\left\{(sCR + 2)^2 - 1\right\} - (sCR + 2)} \tag{1.7}$$

$$= \frac{1}{(sCR + 2)^3 - (sCR + 2)^2 - 2(sCR + 2) + 1} \tag{1.8}$$

$$= \frac{1}{(sCR)^3 - 5(sCR)^2 + 6sCR + 1} \tag{1.9}$$

Problem 1.2. Substitute $s = \jmath 2\pi f, \jmath = \sqrt{-1}$ in (1.9) to obtain $H(f)$. $H(f)$ is known as the frequency response. Plot $|H(f)|$ in python for $-20 < f < 20$,

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given that $R = 1\,k\Omega$ and $C = 10\,\mu F$.

Solution: Type the following code to get Fig. 1.2. You will find that $H(f)$ is a low pass filter.

```
import numpy as np
import matplotlib.pyplot as plt



#Filter Characteristics

R = 1e3; #10K ohm resistance
C = 10e-6;#10 uF capacitance

#Plotting the filter amplitude
    response
T = 0.02;
f_0 = 1/T;
f = np.linspace(-1.5*f_0,1.5*f_0,1
    e2)
s = 1j*2*np.pi*f

den = np.polyval([1,-5, 6, 1],s*C*
    R);
H = 1/den;

plt.plot(f,abs(H))
plt.grid()# minor
plt.xlabel('$f$ (Hz)')
plt.ylabel('$H(f)$')
#Save figure
plt.savefig('../figs/2.2.eps')
plt.show()
```

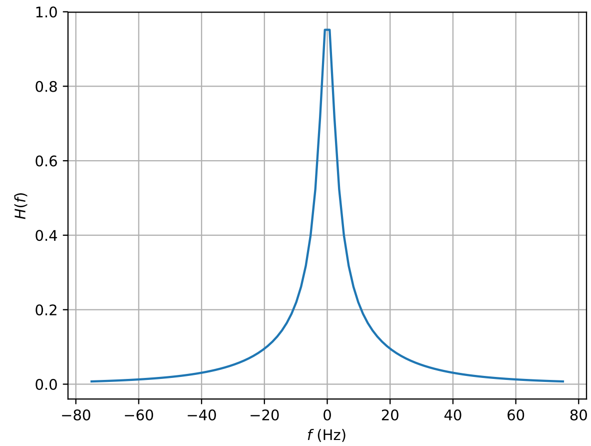

Fig. 1.2: Frequency response of the $R-C$ filter

Problem 1.3. Find the frequency at which $|H(f)|^2 = \frac{1}{2}$. This frequency is known as the 3-dB bandwidth of $H(f)$.

Solution: Substituting sCR = ȷx in (1.9),

$$\left|H(\jmath x)\right| = \frac{1}{\sqrt{2}} \tag{1.10}$$

$$\Rightarrow -jx^3 + 5x^2 + j6x + 1 = \sqrt{2} \tag{1.11}$$

$$\Rightarrow x^2(6 - x^2)^2 + (1 + 5x^2)^2 = 2 \tag{1.12}$$

$$\Rightarrow x^6 + 13x^4 + 46x^2 - 1 = 0 \tag{1.13}$$

Letting $y = x^2$, we obtain the cubic equation

$$y^3 + 13y^2 + 46y - 1 = 0 \tag{1.14}$$

The following script gives the 3 dB bandwidth for the filter H by choosing the real root.

```
import numpy as np
#Filter Characteristics
R = 1e3; #1K ohm resistance
C = 10e-6;#10 uF capacitance

#finding 3 dB bandwidth
    numerically
print(np.sqrt(np.roots([1, 13,
    46, -1]))/(2*np.pi*R*C))
```

This yields the value $f_{3dB} = 2.3395$ Hz.

Problem 1.4. Obtain the 3 dB bandwidth by solving the cubic equation in the previous problem

Solution: In the above, let $y = z - \frac{13}{3}$. Then the equation becomes

$$\Rightarrow z^3 - (31/3)z - 1015/27 = 0 \tag{1.15}$$

This equation has the theoretical solution evaluated by the following script

```
import numpy as np

#Filter Characteristics

R = 1e3; #1K ohm resistance
C = 10e-6;#10 uF capacitance


#finding 3 dB bandwidth
    theoretically
```

```
q = -31/3;
r = -1015/27;

print(np.sqrt((-r/2 + np.sqrt(r
    **2/4 +q**3/27))**(1/3) + (-r/2
    - np.sqrt(r**2/4 +q**3/27))
    **(1/3) - 13/3)/(2*np.pi*R*C))
```

Note that this script gives the same result as the one in the previous problem.

1.2 Circuit Analysis

Problem 1.5. Obtain the expression for $H(s)$ using mesh analysis.

Problem 1.6. Repeat the above exercise using Thevenin's theorem.

Problem 1.7. Repeat the above exercise using Norton's theorem.

Problem 1.8. Repeat the above exercise using $Y-\Delta$ transformation.

Problem 1.9. Obtain all the two port network parameters for the circuit in Fig. 1.1.