Complex Analysis in Probability

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Problem 1. *The PDf of an exponential random variable X is defined as*

$$p_X(t) = \begin{cases} \frac{1}{\gamma} e^{-\frac{t}{\gamma}} & t \ge 0 \\ 0 & t < 0 \end{cases} \tag{1}$$

1) *The* expectation *of the function $g(X)$ is defined as*

$$E\left[g(X)\right] = \int_{\infty}^{\infty} g(t) p_V(t)\, dt \tag{2}$$

Find $E\left[X\right]$.

2) *The cumulative distribution function (CDF) of X is defined as $F_X(t) = \Pr\left(X > t\right)$. Show that the CDF of X is*

$$F_X(t) = \begin{cases} 1 - e^{-\frac{t}{\gamma}} & t \ge 0 \\ 0 & t < 0 \end{cases} \tag{3}$$

3) *The characteristic function of X is defined as $\phi_X(\jmath\omega) = E\left[e^{\jmath\omega X}\right]$. Show that*

$$\phi_X\left(\jmath\omega\right) = \frac{1}{1 - \jmath\omega\gamma} \tag{4}$$

4) *The moment generating function (MGF) of X is defined as $E\left[e^{-sX}\right]$. Show that*

$$M_X\left(s\right) = \frac{1}{1 + s\gamma} \tag{5}$$

Problem 2. *The PDF of X can be* recovered *from the CF using the following formula*

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X\left(\jmath\omega\right) e^{-\jmath\omega x} d\omega \tag{6}$$

since the PDF and CF are governed by the Fourier Transform *relationship.*

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1) *Show that p_X in (6) can be expressed as*

$$p_X(t) = \frac{1}{2\pi\jmath} \int_{-\jmath\infty}^{\jmath\infty} \frac{f(z)}{1 - z\gamma} dz \tag{7}$$

where $f(z) = e^{-zt}$.

2) *Let $z = x + \jmath y$, where x and y are real variables. If $f(z) = u(x, y) + \jmath v(x, y)$, show that*

$$\begin{aligned} u(x, y) &= e^{-xt} \cos(ty) \\ v(x, y) &= -e^{-xt} \sin(ty) \end{aligned} \tag{8}$$

3) *Show that*

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned} \tag{9}$$

The above equations are known as the Cauchy-Reimann *equations and any function f that satisfies the above equations is known as an* Analytic function.

4) *Let C be the unit circle with centre at $\frac{1}{\gamma}$ defined as $C : z = \frac{1}{\gamma} + e^{\jmath\theta}, 0 < \theta < 2\pi$. Show that*

$$\oint_C \frac{dz}{1 - z\gamma} = -\frac{2\pi\jmath}{\gamma} \tag{10}$$

through a change of variables in terms of θ. The integral on the L.H.S. above is known as a contour integral. *Note that the direction of integration is clockwise.*

5) *Let*

$$g(z) = \frac{f(z) - f\left(\frac{1}{\gamma}\right)}{z - \frac{1}{\gamma}} \tag{11}$$

Show that $g(z)$ is analytic for $z \ne \frac{1}{\gamma}$.

6) *An analytic function is also continuous. Show that $|g(z)| < \frac{\epsilon}{\delta}$*

7) *Let* $K : z = \frac{1}{\gamma} + \rho e^{\jmath\theta}, 0 < \theta \le 2\pi$. *If*

$$\oint_K g(z)\, dz \triangleq \sum_k g(z_k)\Delta z_k, \tag{12}$$

show that

$$\left|\oint_K g(z)\right| < 2\pi\epsilon \tag{13}$$

8) *If*

$$\oint_C \frac{f(z)}{1 - z\gamma} dz = \oint_K \frac{f(z)}{1 - z\gamma} dz, \tag{14}$$

show that

$$\oint_C \frac{f(z)}{1 - z\gamma} dz = -\frac{2\pi\jmath}{\gamma} f\left(\frac{1}{\gamma}\right) \tag{15}$$

9) *Let* $D : Re^{\jmath\theta}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *Show that*

$$\lim_{R\to\infty} \int_D \frac{f(z)}{1 - z\gamma} dz = 0 \tag{16}$$

10) *Show that*

$$p_X(t) = \frac{1}{\gamma} e^{-\frac{t}{\gamma}} \quad t \ge 0 \tag{17}$$

11) *Show that*

$$p_X(t) = 0 \quad t < 0 \tag{18}$$

Problem 3. *The Fourier transform of the unit step function* $u(t)$ *is*

$$\int_{-\infty}^{\infty} u(t) e^{\jmath\omega t} = \frac{\delta(\omega)}{2} + \frac{\jmath}{\omega} \tag{19}$$

1) *Show that the CDF of a random variable* X *can be expressed in terms of its CF* $\phi_X(\omega)$ *as*

$$F_X(x) = \frac{1}{2} - \frac{1}{2\pi\jmath} \int_{-\infty}^{\infty} \frac{\phi_X(\omega)}{\omega} e^{-\jmath\omega x}\, d\omega \tag{20}$$

This is known as the Gil-Pelaez formula.

2) *Using the above relation, obtain the CDF in* (3) *using contour integration. You may express* (20) *as*

$$F_X(x) = \frac{1}{2} - \frac{1}{2\pi\jmath} \lim_{\substack{r\to 0 \\ R\to\infty}} \left[\int_{-R}^{r} \frac{\phi_X(\omega)}{\omega} e^{-\jmath\omega x}\, d\omega + \int_{r}^{R} \frac{\phi_X(\omega)}{\omega} e^{-\jmath\omega x}\, d\omega\right] \tag{21}$$

Problem 4. *Repeat the above exercises using the MGF instead of the CDF*