# Geometry through Coordinates

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Contents



*Abstract*—**The focus of this text is on investigating the properties of triangles and circles through coordinate geometry. An early exposure to coordinate geometry allows students to use computers for mathematical visualizations and computing. This book provides an alternative approach to geometry which may be useful to those having difficulties with traditional methods. In the process, almost all the basic concepts of coordinate geometry are covered. Instead of teaching geometry and coordinate geometry as separate subjects in high school, this textbook shows how to develop a holistic approach for teaching math.**

## 1 Medians of a Triangle

### 1.1 Coordinates of Point Dividing a Line in Ratio $k : 1$

**Definition 1.1.** *(*Budhayana's Theorem: *)In Fig. 1.1, ABC is a right angled triangle with $AB = b$ and $BC = c$. Using coordinates, $A, B$ and $C$ can be represented as $(a, b), (0, 0)$ and $(c, 0)$, where $B$ is the origin, $BC$ is on the X-axis and $AB$ is on the Y-axis. Using Budhayana's theorem, $AC = \sqrt{b^2 + c^2}$.*

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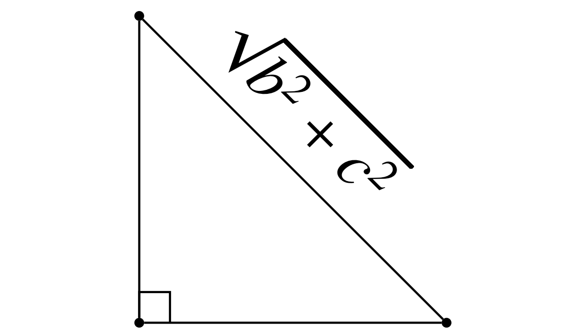


Fig. 1.1: Budhayana's Theorem

**Problem 1.2.** *Let A and B have the coordinates $(a_1, a_2)$ and $(b_1, b_2)$ respectively. Show that the distance between A and B is given by*

$$AB = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \tag{1.1}$$

*Proof.* In Fig. 1.2.1, it is obvious that

$$PB = A_1B_1 = a_1 - b_1 \tag{1.2}$$

$$PA = A_2B_2 = a_2 - b_2 \tag{1.3}$$

Using Budhayana's theorem,

$$AB = \sqrt{PA^2 + PB^2} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \tag{1.4}$$

An alternative visualization of Fig. 1.2.1 is available in 1.2.2, where the line $AB$ is shifted to the origin such that the line becomes $A'O$. This results in

$$OA' = \sqrt{(A'P')^2 + (O'P')^2} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \tag{1.5}$$

Thus, shifting the coordinates by $(b_1, b_2)$ makes the visualization as well as computation easier. This trick will be used quite a lot in this book.

**Problem 1.3.** *In Fig. 1.3, P divides AB in the ratio $k : 1$ such that*

$$\frac{PA}{PB} = k. \tag{1.6}$$

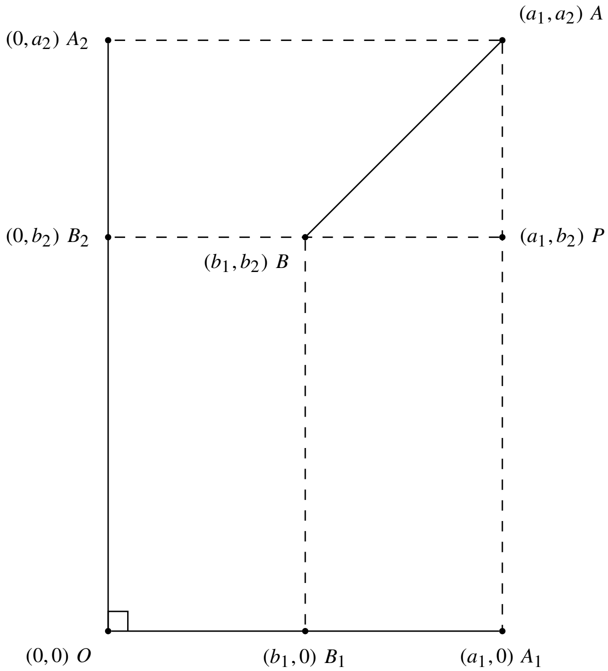


Fig. 1.2.1: Distance formula.

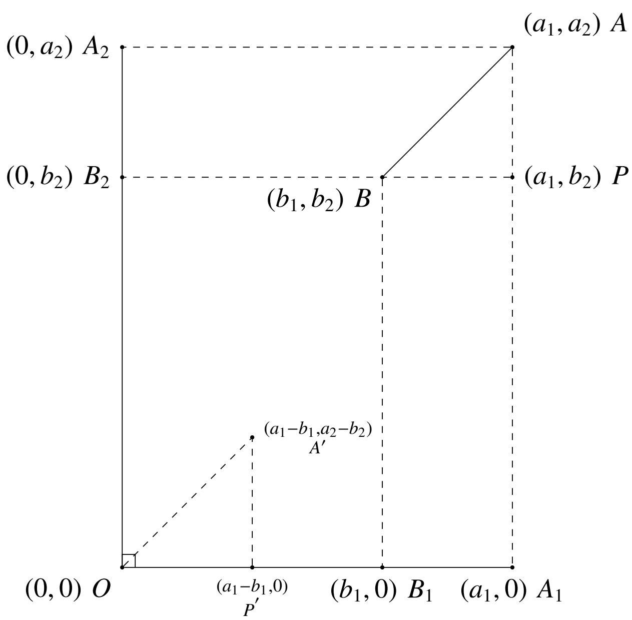


Fig. 1.2.2: Alternative visualization of distance.

*If the coordinates of A and B are $(a_1, a_2)$ and $(b_1, b_2)$ respectively, show that the coordinates of P are*

$$\left(\frac{a_1 + kb_1}{k+1}, \frac{a_2 + kb_2}{k+1}\right) \tag{1.7}$$

*Proof.* From Fig. 1.3, using similar triangles, it is

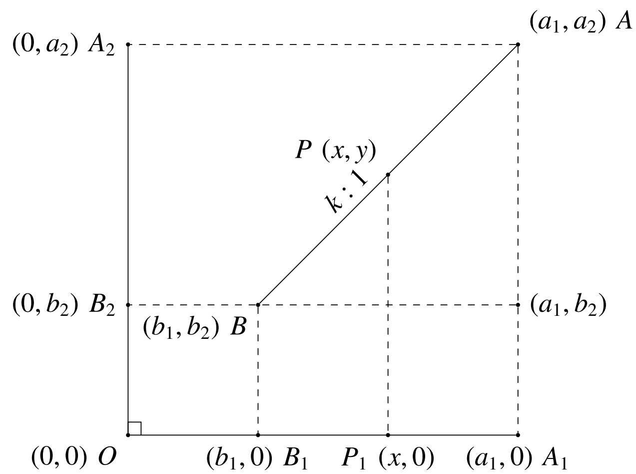


Fig. 1.3: $P$ divides $AB$ in the ratio $k:1$.

obvious that

$$\frac{PB}{PA} = \frac{P_1B_1}{P_1A_1} = \frac{x-b_1}{a_1-x} = k \tag{1.8}$$

Simplifying the above,

$$x - b_1 = k\left(a_1 - x\right) \tag{1.9}$$

$$\Rightarrow (1+k)\,x = ka_1 + b_1 \tag{1.10}$$

$$\Rightarrow x = \frac{ka_1 + b_1}{k+1} \tag{1.11}$$

The proof for $y$ is similar.

### *1.2 Median*

**Definition 1.4.** *The line segment AD in Fig. 1.4 that divides the side BC in two equal halfs (BD = DC) is known as the median.*

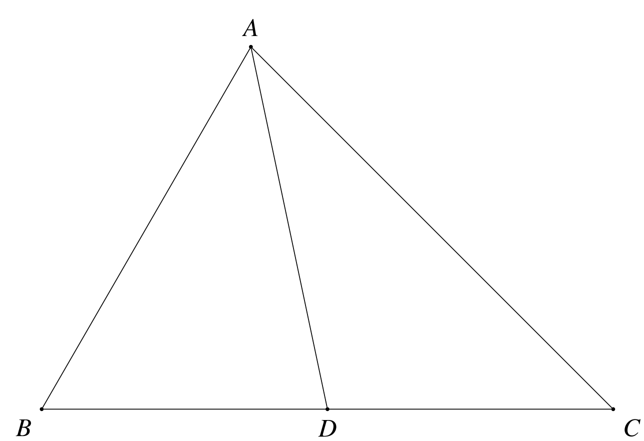


Fig. 1.4: Median

**Problem 1.5.** *The medians $BE$ and $CF$ in Fig. 1.5 intersect at $O$, such that*

$$\begin{aligned}\frac{OB}{OE} &= k_1\\ \frac{OC}{OF} &= k_2\end{aligned} \tag{1.12}$$

*Show that $k_1 = k_2 = 2$.*

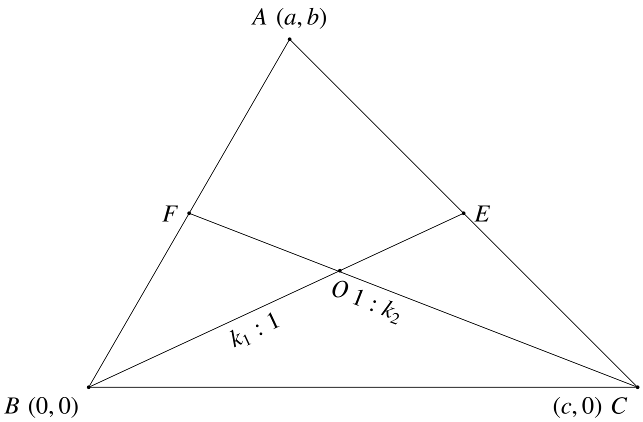


Fig. 1.5: Medians $BE$ and $CF$

*Proof.* Let the coordinates of $A$, $B$ and $C$ be $(a, b)$, $(0, 0)$ and $(c, 0)$ respectively. Using 1.7,

$$E = \begin{pmatrix}\frac{a+c}{2}\\ \frac{b}{2}\end{pmatrix} \tag{1.13}$$

$$F = \begin{pmatrix}\frac{a}{2}\\ \frac{b}{2}\end{pmatrix} \tag{1.14}$$

Since $O$ divides $BE$ in the ratio $k_1 : 1$, from (1.7),

$$O = \frac{k_1 E + B}{k_1 + 1} = \frac{1}{k_1 + 1}\left[k_1\begin{pmatrix}\frac{a+c}{2}\\ \frac{b}{2}\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}\right] \tag{1.15}$$

$$= \frac{k_1}{2(k_1 + 1)}\begin{pmatrix}a + c\\ b\end{pmatrix} \tag{1.16}$$

Similarly, since $O$ divides $CF$ in the ratio $k_2 : 1$,

$$O = \frac{k_2 F + C}{k_2 + 1} = \frac{1}{k_2 + 1}\left[k_2\begin{pmatrix}\frac{a}{2}\\ \frac{b}{2}\end{pmatrix} + \begin{pmatrix}c\\0\end{pmatrix}\right] \tag{1.17}$$

$$= \frac{1}{2(k_2 + 1)}\begin{pmatrix}ak_2 + 2c\\ bk_2\end{pmatrix} \tag{1.18}$$

From (1.16) and (1.18),

$$\frac{k_1}{2(k_1 + 1)}\begin{pmatrix}a + c\\ b\end{pmatrix} = \frac{1}{2(k_2 + 1)}\begin{pmatrix}ak_2 + 2c\\ bk_2\end{pmatrix} \tag{1.19}$$

$$\Rightarrow \frac{k_1}{(k_1 + 1)}\begin{pmatrix}a + c\\ b\end{pmatrix} = \frac{1}{(k_2 + 1)}\begin{pmatrix}ak_2 + 2c\\ bk_2\end{pmatrix} \tag{1.20}$$

By equating the respective coordinates,

$$\frac{k_1(a + c)}{k_1 + 1} = \frac{ak_2 + 2c}{k_2 + 1} \tag{1.21}$$

$$\frac{k_1 b}{k_1 + 1} = \frac{k_2 b}{k_2 + 1} \tag{1.22}$$

From (1.22), it is trivial to obtain $k_1 = k_2$. Substituting this in (1.21) and simplifying,

$$k_1(a + c) = ak_1 + 2c \tag{1.23}$$

$$\Rightarrow k_1 = 2 \tag{1.24}$$

Thus $k_1 = k_2 = 2$.

**Problem 1.6.** *In Fig.1.6,*

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \tag{1.25}$$

*Show that $BE$ and $CF$ are medians.*

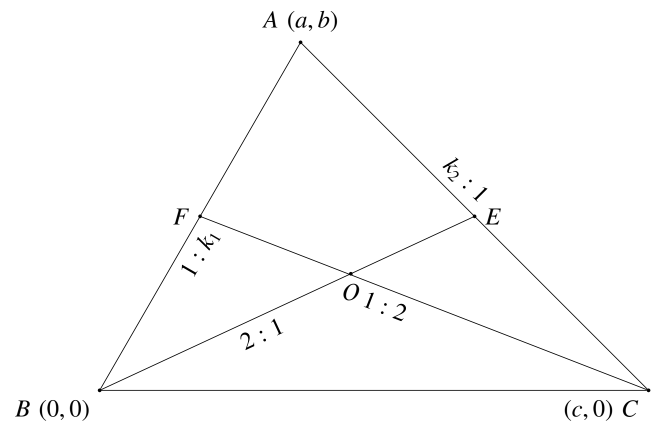


Fig. 1.6: $O$ divides $BE$ and $CF$ in the ratio $2 : 1$.

*Proof.* Let $E$ divide $AC$ in the ratio $k_2 : 1$ and $F$ divide $AB$ in the ratio $k_1 : 1$. Using (1.7),

$$E = \frac{k_2 C + A}{k_2 + 1} = \frac{1}{k_2 + 1}\left[k_2\begin{pmatrix}c\\0\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix}\right] \tag{1.26}$$

$$= \frac{1}{k_2 + 1}\begin{pmatrix}k_2 c + a\\ b\end{pmatrix} \tag{1.27}$$

and

$$F = \frac{k_1 B + A}{k1 + 1} = \frac{1}{k_1 + 1}\left[k_1\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix}\right] \tag{1.28}$$

$$= \frac{1}{k_1 + 1}\begin{pmatrix}a\\b\end{pmatrix} \tag{1.29}$$

Since $O$ divides $BE$ in the ratio $2:1$, using (1.7),

$$O = \frac{2E + B}{3} = \frac{2}{3(k_2+1)}\begin{pmatrix} k_2c + a \\ b \end{pmatrix} \tag{1.30}$$

Also, $O$ divides $CF$ in the ratio $2:1$, hence,

$$O = \frac{2F + C}{3} = \frac{1}{3}\left[\frac{2}{k_1+1}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix}\right] \tag{1.31}$$

$$= \frac{1}{3(k_1+1)}\begin{pmatrix} 2a + (k_1+1)c \\ 2b \end{pmatrix} \tag{1.32}$$

Equating (1.30) and (1.32),

$$\frac{2}{3(k_2+1)}\begin{pmatrix} k_2c + a \\ b \end{pmatrix} = \frac{1}{3(k_1+1)}\begin{pmatrix} 2a + (k_1+1)c \\ 2b \end{pmatrix} \tag{1.33}$$

$$\Rightarrow \frac{2}{(k_2+1)}\begin{pmatrix} k_2c + a \\ b \end{pmatrix} = \frac{1}{(k_1+1)}\begin{pmatrix} 2a + (k_1+1)c \\ 2b \end{pmatrix} \tag{1.34}$$

Equating the respective coordinates,

$$\frac{2(k_2c + a)}{3(k_2+1)} = \frac{2a + (k_1+1)c}{k_1+1} \tag{1.35}$$

$$\frac{2b}{k_2+1} = \frac{2b}{k_1+1} \tag{1.36}$$

From (1.36), $k_1 = k_2$ is trivially obtained. Substituting this in (1.35),

$$\frac{2(k_1c + a)}{3(k_1+1)} = \frac{2a + (k_1+1)c}{k_1+1} \tag{1.37}$$

$$\Rightarrow 2k_1c + 2a = 2a + (k_1+1)c \tag{1.38}$$

$$k_1 = 1 \tag{1.39}$$

after simplification. Thus, $k_1 = k_2 = 1$ and $BE$ and $CF$ are the medians.

**Problem 1.7.** *In Fig. 1.7, let BE and CF be the medians in $\Delta ABC$. Let the line AD pass through O. Show that AD is also a median.*

*Proof.* Substituting $k_2 = 1$ in (1.30),

$$O = \frac{1}{3}\begin{pmatrix} a + c \\ b \end{pmatrix} \tag{1.40}$$

Since $D$ divides $BC$ in the ratio $k_1 : 1$, from (1.7),

$$D = \frac{k_1C + B}{k_1+1} = \frac{1}{k_1+1}\begin{pmatrix} c \\ 0 \end{pmatrix} \tag{1.41}$$

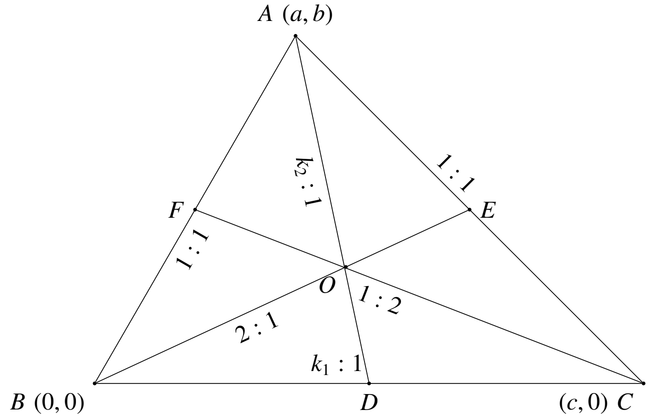


Fig. 1.7: Medians of a triangle are concurrent.

Similarly,

$$O = \frac{k_2D + A}{k_2+1} = \frac{1}{k_2+1}\left[\frac{k_2}{k_1+1}\begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}\right] \tag{1.42}$$

$$= \frac{1}{(k_2+1)(k_1+1)}\begin{pmatrix} k_2c + (k_1+1)a \\ (k_1+1)b \end{pmatrix} \tag{1.43}$$

Equating (1.40) and (1.43),

$$\frac{1}{3}\begin{pmatrix} a + c \\ b \end{pmatrix} = \frac{1}{(k_2+1)(k_1+1)}\begin{pmatrix} k_2c + (k_1+1)a \\ (k_1+1)b \end{pmatrix} \tag{1.44}$$

resulting in

$$\frac{a+c}{3} = \frac{k_2c + (k_1+1)a}{(k_2+1)(k_1+1)} \tag{1.45}$$

$$\frac{b}{3} = \frac{(k_1+1)b}{(k_2+1)(k_1+1)} \tag{1.46}$$

after equating the respective coordinates. From (1.46) $k_2 = 2$ is trivially obtained. Substituting $k_2 = 2$ in (1.45),

$$\frac{a+c}{3} = \frac{2c + (k_1+1)a}{3(k_1+1)} \tag{1.47}$$

$$\Rightarrow (a+c)(k_1+1) = 2c + (k_1+1)a \tag{1.48}$$

$$\Rightarrow k_1 = 1 \tag{1.49}$$

after simplification. Thus $D$ is the midpoint of $BC$ and $AD$ is also a median. All medians intersect at the point $O$. This point is known as the centroid.

## 2 Altitudes of a Triangle

### 2.1 Lines and Angles

**Problem 2.1.** *In Fig. 2.1, the line passing through the points $A, B, C$ makes an angle $\theta$ with the X-axis.*

*Show that the equation of the line is*

$$y = mx + c, \tag{2.1}$$

*where* $(x, y)$ *is any point on the line and* $m = \tan\theta$. $m$ *is also known as the slope of the line.*

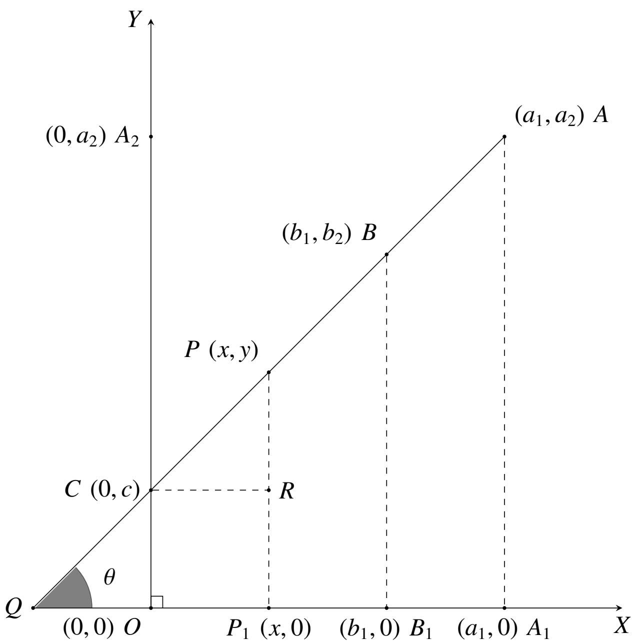


Fig. 2.1: The straight line.

*Proof.* Let $P = (x, y)$ be any point on the line. From the figure, is is obvious that

$$\frac{PR}{CR} = \frac{y - c}{x} = \tan\theta = m \tag{2.2}$$

Rearranging terms,

$$y = mx + c \tag{2.3}$$

**Problem 2.2.** *Show that the equation of the line in Fig. 2.1 can also be expressed as*

$$\frac{y - b_2}{x - b_1} = \frac{a_2 - b_2}{a_1 - b_1} \tag{2.4}$$

*Proof.* Looking at the line segments $AB$ and $PB$ and following the approach in Problem 2.1,

$$\tan\theta = \frac{a_2 - b_2}{a_1 - b_1} = \frac{y - b_2}{x - b_1} \tag{2.5}$$

**Problem 2.3.** *Show that the equation*

$$px + qy + r = 0 \tag{2.6}$$

*is that of a straight line.*

*Proof.* The above equation can be rearranged as

$$y = -\frac{p}{q}x - \frac{r}{q} \tag{2.7}$$

which is the same as (2.1) with $m = -\frac{p}{q}$ and $c = -\frac{r}{q}$.

**Problem 2.4.** *The point of intersection of the lines*

$$p_1x + q_1y + r_1 = 0 \tag{2.8}$$
$$p_2x + q_2y + r_2 = 0 \tag{2.9}$$

*is*

$$x = -\frac{\begin{vmatrix} r_1 & q_1 \\ r_2 & q_2 \end{vmatrix}}{\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}} \tag{2.10}$$

$$y = -\frac{\begin{vmatrix} p_1 & r_1 \\ p_2 & r_2 \end{vmatrix}}{\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}} \tag{2.11}$$

**Problem 2.5.** *Show that angle between the lines*

$$y = m_1x + c_1 \tag{2.12}$$
$$y = m_2x = c_2 \tag{2.13}$$

*is*

$$\theta = \tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2} \tag{2.14}$$

*Proof.* If $\theta_1$ and $\theta_2$ are the angles made by the two lines with the $X$-axis respectively, the angle between the two is $\theta = \theta_1 - \theta_2$. Thus, using the trigonometric identity,

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2} \tag{2.15}$$

$$= \frac{m_1 - m_2}{1 + m_1m_2} \tag{2.16}$$

**Problem 2.6.** *Show that the angle between two lines is* $90^\circ$ *if* $m_1m_2 = -1$.

### 2.2 Altitude

**Definition 2.7.** *In* $\Delta ABC$ *Fig. 2.7,* $AD \perp BC$ *is known as the altitude.*

**Problem 2.8.** *In Fig. 2.8, the altitudes AD and BE meet at O. CF is the line obtained by extending CO to meet AB at F. Show that CF is also an altitude of* $\Delta ABC$.

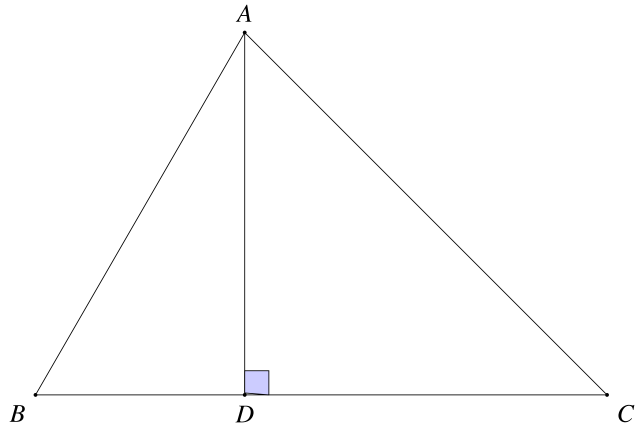


Fig. 2.7: Altitude $AD$.

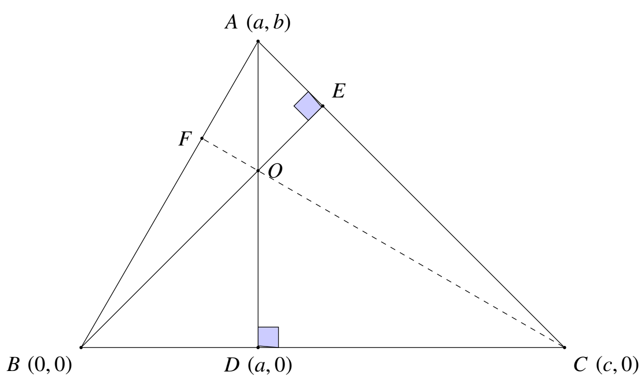


Fig. 2.8: Altitudes $AD$ and $BE$ meet at $O$.

*Proof.* The equation for $AD$ is

$$x = a \tag{2.17}$$

The slope of $AC$ is

$$\frac{b}{a-c}. \tag{2.18}$$

Thus, using Problem 2.6, the slope of $CE$ is

$$m_{CE} = \frac{c-a}{b} \tag{2.19}$$

Using (2.1) and (2.19), the equation of $CE$ is

$$y = m_{CE}(x-0) = \frac{c-a}{b}x \tag{2.20}$$

since $O$ is the intersection of $AD$ and $CE$, from (2.20) and (2.17), its coordinates are

$$\left(a, \left(\frac{c-a}{b}\right)a\right) \tag{2.21}$$

From (2.21), the slope of $CO$ (and $CF$) is

$$m_{CF} = \frac{\left(\frac{c-a}{b}\right)a}{a-c} = -\frac{a}{b} \tag{2.22}$$

From Fig. 2.8, the slope of $AB$ is

$$m_{AB} = \frac{b}{a} \tag{2.23}$$

Since

$$m_{AB}m_{CF} = -1, \tag{2.24}$$

using Problem 2.6, $AB \perp CF$ and $CF$ is an altitude of $\Delta ABC$.

## 3 The Circle

### 3.1 Definitions

**Definition 3.1.** (Circle): *Fig. 3.1 shows a circle with centre O and radius r, where each point on the circle is at the same distance r from the centre O.*

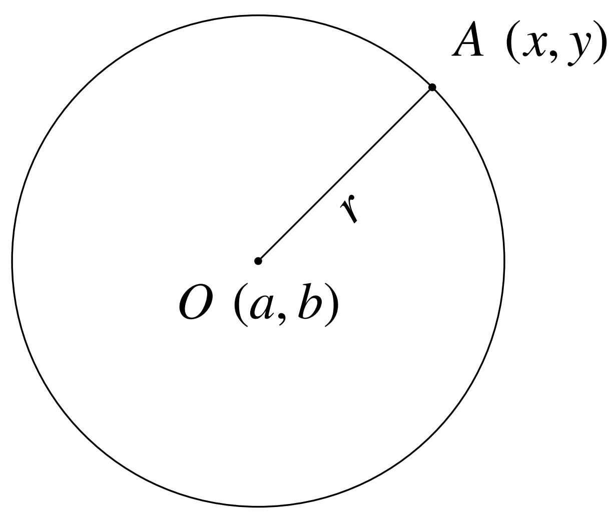


Fig. 3.1: Circle with radius $r$.

**Problem 3.2.** *Show that the equation of the circle is*

$$(x-a)^2 + (y-b)^2 = r^2 \tag{3.1}$$

*Proof.* Since each point $(x, y)$ on the circle is at a distance $r$ from the centre $(a, b)$, using (1.1), (3.1) is obtained.

**Definition 3.3.** (Tangent)*: In Fig. 3.3, the line BX that touches the circle at exactly one point C is known as the tangent.*

### 3.2 Tangent

**Problem 3.4.** *In Fig. 3.3, show that the tangent $BX \perp OC$.*

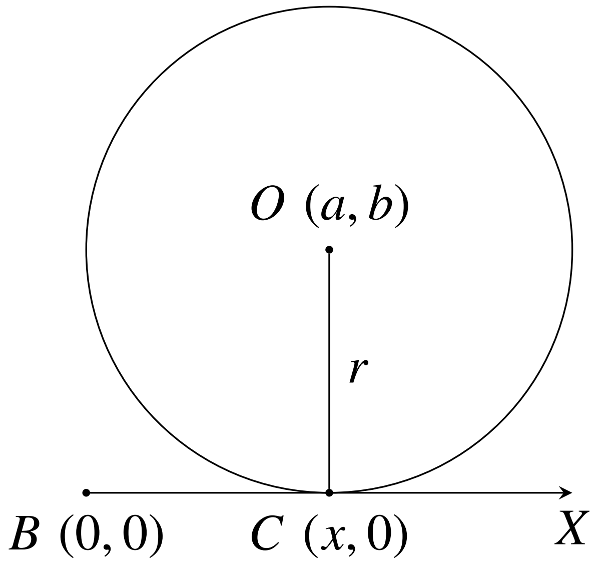


Fig. 3.3: Tangent to the circle.

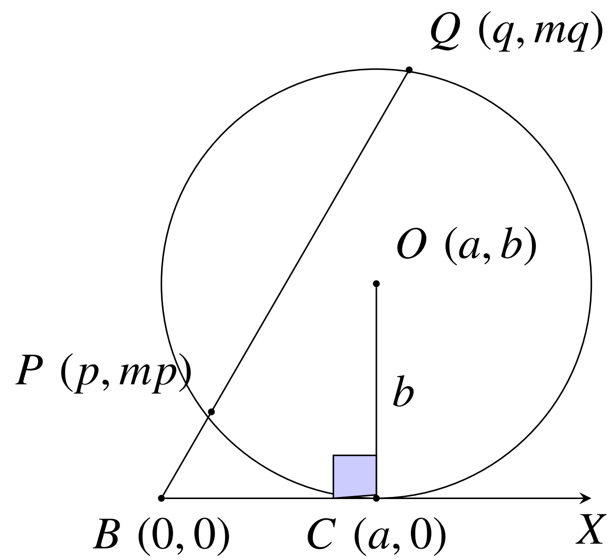


Fig. 3.5: $BP.BQ = BC^2$.

*Proof.* Since $C$ lies on the circle and is at distance $r$ from the centre $O$, using (1.1),

$$(x-a)^2 + b^2 = r^2 \tag{3.2}$$

$$\Rightarrow x = a \pm \sqrt{r^2 - b^2} \tag{3.3}$$

By the definition of a tangent, the point $C$ should be unique. Thus, in (3.3), $x = a$ and $r = b$. By definition of the coordnate system, this implies that $\Delta OCB$ is right angled and $BX \perp OC$.

**Problem 3.5.** *In Fig. 3.5, $BX$ is a tangent to the circle centred at $O$ and touches it at $C$. $P$ and $Q$ are two points on the circle. Show that*

$$BP.BQ = BC^2 \tag{3.4}$$

*Proof.* Let $m$ be the slope of the line $BQ$. Then the coordinates of $P$ and $Q$ can be represented as $(p, mp)$ and $(q, mq)$ respectively. The radius of the circle is $b$. Using (3.1) for $P$ and $Q$,

$$(p-a)^2 + (mp-b)^2 = b^2 \tag{3.5}$$

$$\Rightarrow a^2 - 2ap + m^2p^2 = 2mpb \tag{3.6}$$

and

$$\Rightarrow a^2 - 2aq + m^2q^2 = 2mqb \tag{3.7}$$

Dividing (3.6) by (3.7),

$$\frac{a^2 - 2ap + p^2\left(1+m^2\right)}{a^2 - 2aq + q^2\left(1+m^2\right)} = \frac{p}{q} \tag{3.8}$$

which, after cross multiplying, results in

$$q\left(a^2 - 2ap + p^2\left(1+m^2\right)\right) = p\left(a^2 - 2aq + q^2\left(1+m^2\right)\right) \tag{3.9}$$

and can be simplified as

$$a^2(p-q) = pq(p-q)\left(1+m^2\right) \tag{3.10}$$

$$\Rightarrow a^2 = pq\left(1+m^2\right) \tag{3.11}$$

$$\Rightarrow a^2 = p\sqrt{(1+m^2)}.q\sqrt{(1+m^2)} \tag{3.12}$$

$$\Rightarrow BC^2 = BP.BQ, \tag{3.13}$$

using (1.1) for obtaining the length of $BP$ and $BQ$.

### *3.3 Semi-circle*

**Problem 3.6.** *Fig. 3.6 shows a semi circle. $BC = 2r$, where $r$ is the radius of the circle is known as the diameter. Show that $\angle BAC = 90^\circ$.*

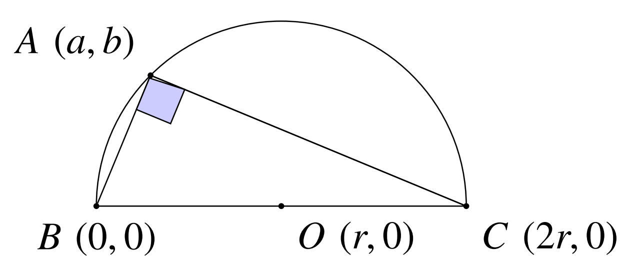


Fig. 3.6: $\angle BAC = 90^\circ$.

*Proof.* From (3.1),

$$(a-r)^2+b^2=r^2. \tag{3.14}$$

From (1.1),

$$\begin{aligned} AB^2+BC^2 &= \left(a^2+b^2\right)+(a-2r)^2+b^2 && (3.15)\\ &= 2\left(a^2-2ar+b^2+r^2\right)+2r^2 && (3.16)\\ &= 2\left[(a-r)^2+b^2\right]+2r^2 && (3.17)\\ &= 4r^2 \quad \text{(from (3.14) )} && (3.18)\\ &= BC^2 && (3.19)\end{aligned}$$

From Budhayana's thorem, $\angle ABC = 90^\circ$.

*3.4 More Properties*

**Problem 3.7.** *In Fig. 3.7, show that*

$$\angle BOC = 2\angle BAC \tag{3.20}$$

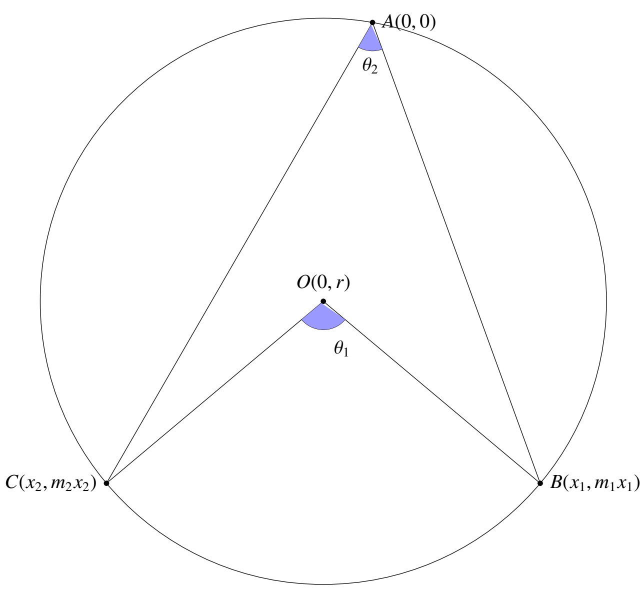


Fig. 3.7: $\theta_1 = 2\theta_2$.

**Solution:** In Fig. 3.7, $O$ lies on the $y$-axis. Hence, t he radius of the circle is $r$. Since $C$ is a point on the circle, using the equation of the circle,

$$\begin{aligned} x_1^2+(m_1x-r)^2 &= r^2 && (3.21)\\ \Rightarrow x_1\left(1+m_1^2\right) &= 2m_1r && (3.22)\\ \Rightarrow r &= \frac{x_1\left(1+m_1^2\right)}{2m_1} && (3.23)\end{aligned}$$

Since a similar property holds for the point $B$,

$$\Rightarrow r = \frac{x_1\left(1+m_1^2\right)}{2m_1} = \frac{x_2\left(1+m_2^2\right)}{2m_2} \tag{3.24}$$

From (2.14),

$$\begin{aligned} \tan\theta_1 &= \frac{\frac{m_1x_1-r}{x_1}-\frac{m_2x_2-r}{x_2}}{1+\frac{m_1x_1-r}{x_1}\frac{m_2x_2-r}{x_2}} && (3.25)\\ &= \frac{m_1-\frac{r}{x_1}-\left(m_2-\frac{r}{x_2}\right)}{1+\left(m_1-\frac{r}{x_1}\right)\left(m_2-\frac{r}{x_2}\right)} && (3.26)\end{aligned}$$

From (3.24),

$$\begin{aligned} m_1-\frac{r}{x_1} &= \frac{m_1^2-1}{2m_1} && (3.27)\\ m_2-\frac{r}{x_2} &= \frac{m_2^2-1}{2m_2} && (3.28)\end{aligned}$$

Hence,

$$\begin{aligned} \tan\theta_1 &= \frac{\frac{m_1^2-1}{2m_1}-\frac{m_2^2-1}{2m_2}}{1+\frac{m_1^2-1}{2m_1}\cdot\frac{m_2^2-1}{2m_2}} && (3.29)\\ &= \frac{2(m_1-m_2)(1+m_1m_2)}{(4m_1m_2)+\left(m_1^2-1\right)\left(m_2^2-1\right)} && (3.30)\end{aligned}$$

Again, using (2.14) for $\theta_2$,

$$\begin{aligned} \tan\theta_2 &= \frac{m_1-m_2}{1+m_1m_2} && (3.31)\\ \Rightarrow \tan 2\theta_2 &= \frac{2\tan\theta_2}{1-\tan^2\theta_2} && (3.32)\\ &= \frac{2\frac{m_1-m_2}{1+m_1m_2}}{1-\left(\frac{m_1-m_2}{1+m_1m_2}\right)^2} && (3.33)\\ &= \frac{2(m_1-m_2)(1+m_1m_2)}{(1+m_1m_2)^2-(m_1-m_2)^2} && (3.34)\\ &= \tan\theta_1 && (3.35)\end{aligned}$$

after simplification. □

**Problem 3.8.** *In Fig. 3.8, show that*

$$\angle PBC = \angle BAC. \tag{3.36}$$

**Solution:** The slopes of $AB, BC, CA$ are, respectively,

$$\begin{aligned} m_{AB} &= m_1 && (3.37)\\ m_{BC} &= m_2 = \tan\theta_1 && (3.38)\\ m_{CA} &= \frac{m_2x_2-m_1x_1}{m_2-m_1} && (3.39)\end{aligned}$$

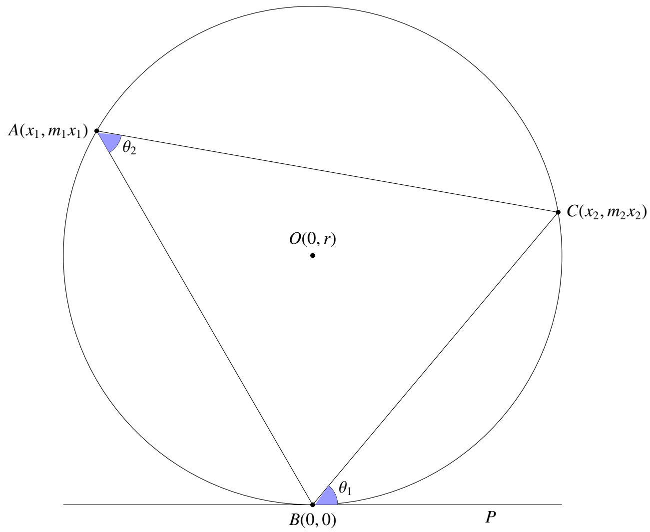


Fig. 3.8: $\theta_1 = \theta_2$

From (2.14),

$$\tan\theta_2 = \frac{\frac{m_2x_2 - m_1x_1}{m_2 - m_1} - m_1}{1 + m_1 \cdot \frac{m_2x_2 - m_1x_1}{m_2 - m_1}} \tag{3.40}$$

$$= \frac{x_2(m_2 - m_1)}{x_2(1 + m_1m_2) - x_1\left(1 + m_1^2\right)} \tag{3.41}$$

$$= \frac{(m_2 - m_1)}{(1 + m_1m_2) - \frac{x_1}{x_2}\left(1 + m_1^2\right)} \tag{3.42}$$

after simplification. From (3.24),

$$\frac{x_1}{x_2} = \frac{m_1}{m_2} \cdot \frac{\left(1 + m_2^2\right)}{\left(1 + m_1^2\right)} \tag{3.43}$$

Substituting (3.43) in (3.42),

$$\tan\theta_2 = \frac{(m_2 - m_1)}{(1 + m_1m_2) - \frac{m_1}{m_2} \cdot \frac{\left(1+m_2^2\right)}{\left(1+m_1^2\right)}\left(1 + m_1^2\right)} \tag{3.44}$$

$$= \frac{(m_2 - m_1)}{(1 + m_1m_2) - \frac{m_1}{m_2}\left(1 + m_2^2\right)} \tag{3.45}$$

$$= m_2 = \tan\theta_1 \tag{3.46}$$

after simplification $\square$.