Digital Modulation Techniques

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Contents

***Abstract*—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.**

1 BPSK

Problem 1. The *signal constellation diagram* for BPSK is given by Fig. 1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

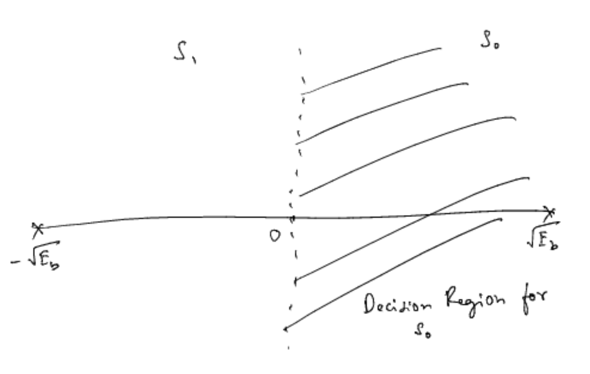

Fig. 1

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Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1}$$
$$y|s_1 = -\sqrt{E_b} + n \tag{2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Problem 2. From Fig. 1 obtain a decision rule for BPSK

Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{3}$$

Problem 3. Repeat the previous exercise using the MAP criterion.

Problem 4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0|s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{4}$$
$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{5}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{6}$$
$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{7}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

Problem 5. The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{8}$$

and the complementary error function is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt. \tag{9}$$

Show that

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{10}$$

Problem 6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt

def qfunc(x):
        return 0.5*mp.erfc(x/mp.sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
simlen = int(1e5)
#Simulated BER declaration
err = []
#Analytical BER declaration
ber = []

#for SNR 0 to 10 dB
for i in range(0,snrlen):
        #Generating AWGN, 0 mean unit variance
        noise = np.random.normal(0,1,simlen)
        #from dB to actual SNR
        snr = 10**(0.1*snrdb[i])
        #Received symbol in baseband
        rx = mp.sqrt(snr) + noise
        #storing the index for the received symbol
        #in error
        err_ind = np.nonzero(rx < 0)
        #calculating the total number of errors
        err_n = np.size(err_ind)
        #calcuating the simulated BER
        err.append(err_n/simlen)
        #calculating the analytical BER
        ber.append(qfunc(mp.sqrt(snr)))

plt.semilogy(snrdb.T,ber,label='Analysis')
plt.semilogy(snrdb.T,err,'o',label='Sim')
plt.xlabel('SNR$\\left(\\frac{E_b}{N_0}\\right)$')
plt.ylabel('$P_e$')
plt.legend()
plt.grid()
plt.savefig('../figs/bpsk_ber.eps')
plt.show()
```

yields Fig. 2

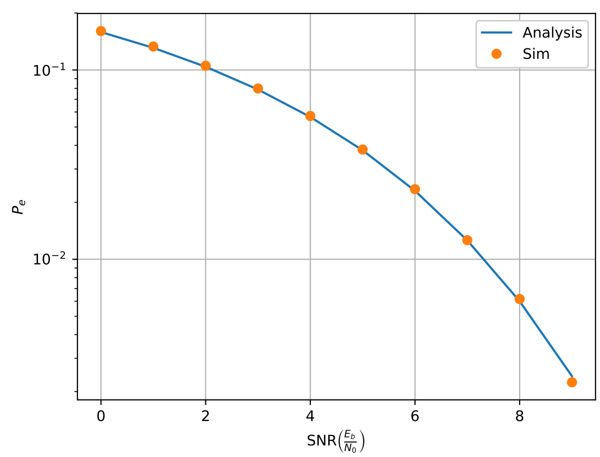

Fig. 2

Problem 7. Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{11}$$

2 Coherent BFSK

Problem 8. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 3. Obtain the equations for the received symbols.

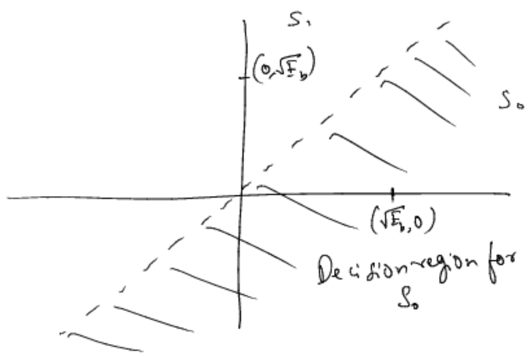

Fig. 3

Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{12}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{13}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Problem 9. Obtain a decision rule for BFSK from Fig. 3.

Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{14}$$

Definition 2.1. The joint PDF of X, Y is given by

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right\}\right] \tag{15}$$

where

$$\mu_x = E[X], \sigma_x^2 = \text{var}(X), \rho = \frac{E\left[(X-\mu_x)\left(Y-\mu_y\right)\right]}{\sigma_x\sigma_y}. \tag{16}$$

Problem 10. For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{17}$$

Use (15) in (17) to obtain (14).

Solution: According to the MAP criterion, assuming equiprobably symbols,

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{18}$$

Problem 11. Derive and plot the probability of error. Verify through simulation.

Solution: Given that s_0 was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{19}$$

From (14), the probability of error is given by

$$P_e = \Pr(y_1 < y_2|s_0) = \Pr\left(\sqrt{E_b} + n_1 < n_2\right) \tag{20}$$

$$= \Pr\left(n_2 - n_1 > \sqrt{E_b}\right) \tag{21}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, N_0)$. Thus,

$$P_e = \Pr\left(\sqrt{N_0}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \tag{22}$$

$$\implies P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{23}$$

where $w \sim \mathcal{N}(0,1)$. The following code plots the BER curves in Fig. 4

```
from __future__ import division
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt


#the ber expression is sqrt(E_b)+
   n1-n2<0

def qfunc(x):
        return 0.5*mp.erfc(x/mp.
           sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
simlen = int(1e5)
#Simulated BER declaration
err = []
```

```
#Analytical BER declaration
ber = []
noise1 = np.random.normal(0,1,
    simlen)
noise2=np.random.normal(0,1,simlen
    )
#for SNR 0 to 10 dB
for i in range(0,snrlen):
        #Generating AWGN, 0 mean
            unit variance

        #from dB to actual SNR
        snr = 10**(0.1*snrdb[i])
        #Received symbol in
            baseband
        y1 = mp.sqrt(2*snr) +
            noise1
        y2=noise2
        #storing the index for the
            received symbol
        #in error
        err_ind = np.nonzero(y1 <
            y2)
        #calculating the total
            number of errors
        err_n = np.size(err_ind)
        #calcuating the simulated
            BER
        err.append(err_n/simlen)
        #calculating the
            analytical BER
        ber.append(qfunc(mp.sqrt(
            snr)))

plt.semilogy(snrdb.T,ber,label='
    Analysis')
plt.semilogy(snrdb.T,err,'o',label
    ='Sim')
plt.xlabel('SNR$\\left(\\frac{E_b
    }{N_0}\\right)$')
plt.ylabel('$P_e$')
plt.legend()
plt.grid()
plt.savefig('../figs/bfsk_ber.eps'
    )
plt.show()
```

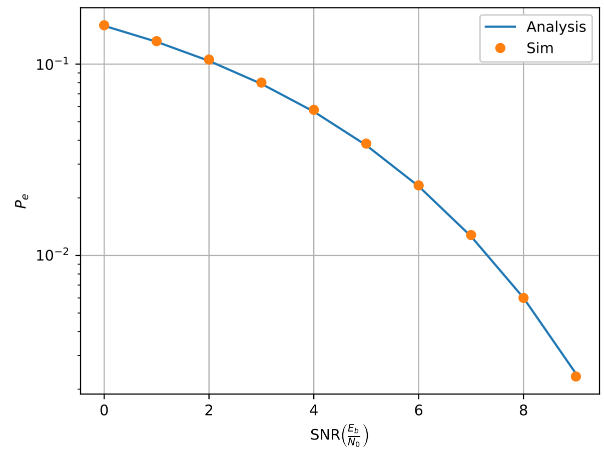

Fig. 4

3 QPSK

Let

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \tag{24}$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}, \tag{25}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_s} \end{pmatrix}, \tag{26}$$

$$E[\mathbf{n}] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{27}$$

Problem 12. Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|y_2| < y_1 \tag{28}$$

Problem 13. Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[(X - \mu_x)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \tag{29}$$

X and Y are said to be uncorrelated if $\rho = 0$

Problem 14. Show that if X and Y are uncorrelated Verify this numerically.

Problem 15. Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x) p_Y(y)$.

Problem 16. Show that $X, Y \sim \mathcal{N}(0, N_0)$.

Problem 17. Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr\left(X < \sqrt{E_s}, Y < \sqrt{E_s}\right). \tag{30}$$

Problem 18. Show that

$$\Pr\left(X < \sqrt{E_s}, Y < \sqrt{E_s}\right) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \quad (31)$$

Problem 19. Verify the above through simulation.

Solution: This is shown in Fig. 5 through the following code.

```
from __future__ import division
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt




def qfunc(x):
        return 0.5*mp.erfc(x/np.
           sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
simlen = int(1e5)
#Simulated BER declaration
err = []
#Analytical BER declaration
ber = []
temp=0
noise1 = np.random.normal(0,1,
   simlen)
noise2=np.random.normal(0,1,simlen
   )

#for SNR 0 to 10 dB
for i in range(0,snrlen):
        snr = 10**(0.1*snrdb[i])
                   #Received symbol
           in baseband
        rx = mp.sqrt(2*snr) +
           noise1
        ry = noise2
        temp=0
        for j in range (0,len(rx))
           :
            if ((rx[j]>ry[j]) and
               (rx[j]>-ry[j])):
                temp=temp+1
        #calculating the total
           number of errors
        #err_n = np.size(err_ind)
        #calcuating the simulated
           BER
        err.append(temp/simlen)
        #calculating the
           analytical BER
        ber.append((1-qfunc(mp.
           sqrt(snr)))**2)

plt.semilogy(snrdb.T,ber,label='
   Analysis')
plt.semilogy(snrdb.T,err,'o',label
   ='Sim')
plt.xlabel('SNR$\\left(\\frac{E_b
   }{N_0}\\right)$')
plt.ylabel('$P_e$')
plt.legend()
plt.grid()
plt.savefig('../figs/qpsk.eps')
plt.show()
```

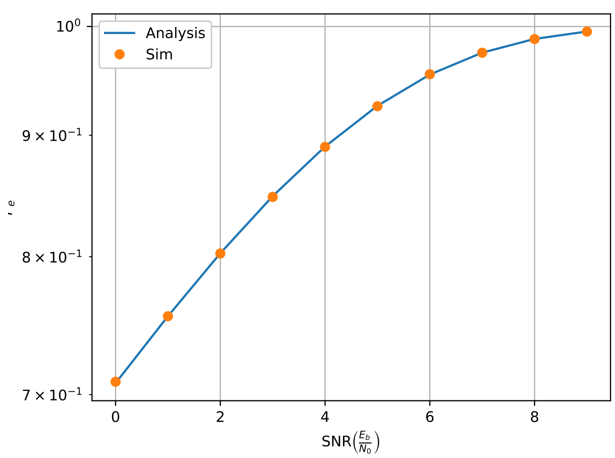

Fig. 5

Problem 20. Modify the above script to obtain the probability of symbol error.

4 M-PSK

Consider a system where $\mathbf{s}_i = \begin{pmatrix}\cos\left(\frac{2\pi i}{M}\right)\\ \cos\left(\frac{2\pi i}{M}\right)\end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{y}|s_0 = \begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}\sqrt{E_s}+n_1\\ n_2\end{pmatrix} \tag{32}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Problem 21. Substituting

$$y_1 = R\cos\theta \tag{33}$$

$$y_2 = R\sin\theta \tag{34}$$

show that the joint pdf of R, θ is

$$p(R,\theta) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{35}$$

Problem 22. Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\, e^{-(V-\alpha)^2}\, dV = 0 \tag{36}$$

$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\, dV = \sqrt{\pi} \tag{37}$$

Problem 23. Using the above, show that

$$\int_0^\infty V\exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV = e^{-\gamma\sin^2\theta}\sqrt{\gamma\pi}\cos\theta \tag{38}$$

for large values of γ.

Problem 24. Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\, d\theta \tag{39}$$

Problem 25. Show that

$$P_{e|\mathbf{s}_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_o}\right)}\sin\frac{\pi}{M}\right) \tag{40}$$

Problem 26. Verify the SER through simulation.