Matrix Analysis through Python

G V V Sharma*

Contents

***Abstract*—The manual introduces a system of equations with no solution, which is solved using Moore-Penrose pseudo inverse in Python. An alternative method for obtaining the pseudo inverse using SVD is also employed. In the process, all basic concepts in matrix analysis like eigenvalues, eigenvectors, orthogonality, Gram-Schmidt orthogonalization, symmetric matrices and SVD are covered.**

1 Least Squares

1.1 Problem

Problem 1. Sketch the vectors

$$\mathbf{a}_1 = (1,1,1)^T, \mathbf{a}_2 = (0,1,2)^T, \mathbf{b} = (6,0,0)^T \tag{1.1}$$

in the 3-D plane.

Problem 2. Find x_1, x_2 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \tag{2.1}$$

geometrically.

Problem 3. Solve the matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.1}$$

where $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2]$ using row reduction. Comment.

1.2 Solution using Python

Problem 4. Run the following Python code and comment on the output for different values of $\mathbf{x}$

```
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
from numpy import linalg as LA

#Creating matrix
A=np.matrix('1 0; 1 1; 1 2')
#Creating vector
b=np.matrix('6; 0; 0')
#P = inv((A'A)A'
P=np.dot(inv(np.dot(np.transpose(A
    ), A)), np.transpose(A))
#x_ls=Pb
x_ls=np.dot(P, b)
x=np.matrix('5; -5')
#||b-Ax_ls||
exact_ls_metric=(LA.norm(b-np.dot(
    A, x_ls)))**2
#||b-Ax||
random_ls_metric=(LA.norm(b-np.dot
    (A, x)))**2
print(x_ls)
print(x)
print(exact_ls_metric)
print(random_ls_metric)
```

Problem 5. Compare the results obtained by typing the following code with the results in the previous problem.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. This work was funded by the PMMMNMTT, MHRD, Govt. of India.

```python
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
from numpy import linalg as LA

A=np.matrix('1 0; 1 1; 1 2')
b=np.matrix('6; 0; 0')

#SVD, A=USV
U, s, V=LA.svd(A)
#Find size of A
mn=A.shape
#Creating the singular matrix
S = np.zeros(mn)
Sinv = S.T
S[:2,:2] = np.diag(s)
#Verifying the SVD, A=USV
print(U.dot(S).dot(V))
#Inverting s
sinv = 1./s
#Inverse transpose of S
Sinv[:2,:2] = np.diag(sinv)
print(Sinv)
#Moore-Penrose Pseudoinverse
Aplus = V.T.dot(Sinv).dot(U.T)
#Least squares solution
x_ls = Aplus.dot(b)
#
print(x_ls)
```

Problem 6. Type the following code in Python and run. Comment.

```python
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
from numpy import linalg as LA

#Creating matrices
A=np.matrix('1 0; 1 1; 1 2')
b=np.matrix('6; 0; 0')
#Eigenvalue decompostion of A'A
Dv, Pv=LA.eig(A.T.dot(A))
#Eigenvalue decompostion of AA'
Du, Pu=LA.eig(A.dot(A.T))
#Singular values of A
Stemp=np.sqrt(Dv)
#QR Decomposition to get U and V
V, Rv=LA.qr(Pv)
U, Ru=LA.qr(Pu)
#SVD, A=USV
U_1, s, V_1=LA.svd(A)
print(V)
print(V_1)
print(U)
print(U_1)
print(s)
print(Stemp)
```

Let

$$g(\mathbf{x}) = \|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2 \tag{6.1}$$

Problem 7. Using calculus, minimize $g(\mathbf{x})$.

Problem 8. Find $(A^T A)^{-1} A^T b$

2 Matrix Analysis

Verify your results through Python, wherever possible.

2.1 Eigenvalues and Eigenvectors

For any square matrix $\mathbf{G}$, if

$$\mathbf{G}\mathbf{x} = \lambda\mathbf{x}, \tag{8.1}$$

λ is known as the *eigenvalue* and $\mathbf{x}$ is the corresponding *eigenvector*.

Let

$$\mathbf{G} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \tag{8.2}$$

Problem 9. Show that the eigenvalues of $\mathbf{G}$ are obtained by solving the equation

$$f(\lambda) = |\lambda\mathbf{I} - G| = 0 \tag{9.1}$$

Note that (9.1) is known as the *characteristic equation*. $f(\lambda)$ is known as the characteristic polynomial.

Problem 10. Obtain the eigenvalues and eigenvectors of $\mathbf{G}$.

Problem 11. Find $f(\mathbf{G})$. This is known as the *Cayley-Hamilton Theorem.*

Problem 12. Stack the eigenvalues of $\mathbf{G}$ in a diagonal matrix $\mathbf{\Lambda}$ and the corresponding eigenvectors in a matrix $\mathbf{F}$. Find $\mathbf{F}\mathbf{\Lambda}\mathbf{F}^{-1}$. This is known as *Eigenvalue Decomposition*

2.2 Symmetric Matrices

Let

$$\mathbf{C} = \begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix} \tag{12.1}$$

Note that $\mathbf{C} = \mathbf{C}^T$. Such matrices are known as *symmetric matrices*.

Problem 13. Find $\mathbf{P}$ such that $\mathbf{C} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

Problem 14. Find $\mathbf{PP}^T$ and $\mathbf{P}^T\mathbf{P}$. $\mathbf{P}$ is known as an *orthogonal matrix*.

Let

$$\mathbf{B} = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix} \tag{14.1}$$

Problem 15. Find $\mathbf{B}^T\mathbf{B}$ and $\mathbf{BB}^T$

Note that $\mathbf{C} = \frac{1}{9}\left(\mathbf{BB}^T\right)$.

Problem 16. Obtain the eigenvalues and eigenvectors of $\mathbf{B}^T\mathbf{B}$

Problem 17. Verify eigenvalue decomposition and Cayley-Hamilton theorem for $\mathbf{B}^T\mathbf{B}$.

2.3 Orthogonality

Let $\mathbf{v}_1, \mathbf{v}_2$ be the columns of $\mathbf{C}$.

Problem 18. Obtain $\mathbf{u}_1, \mathbf{u}_2$ from $\mathbf{v}_1, \mathbf{v}_2$ through the following equations.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \tag{18.1}$$

$$\hat{\mathbf{u}}_2 = \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\,\mathbf{u}_1 \tag{18.2}$$

$$\mathbf{u}_2 = \frac{\hat{\mathbf{u}}_2}{\|\hat{\mathbf{u}}_2\|} \tag{18.3}$$

This procedure is known as Gram-Schmidt orthogonalization.

Problem 19. Stack the vectors $\mathbf{u}_1, \mathbf{u}_2$ in columns to obtain the matrix $\mathbf{Q}$. Show that $\mathbf{Q}$ is orthogonal.

Problem 20. From the Gram=Schmidt process, show that $\mathbf{C} = \mathbf{QR}$, where $\mathbf{R}$ is an upper triangular matrix. This is known as the $\mathbf{Q} - \mathbf{R}$ decomposition.

2.4 Singular Value Decomposition

Problem 21. Find an orthonormal basis for $\mathbf{B}^T\mathbf{B}$ comprising of the eigenvectors. Stack these orthonormal eigenvectors in a matrix $\mathbf{V}$. This is known as *Orthogonal Diagonalization*.

Problem 22. Find the singular values of $\mathbf{B}^T\mathbf{B}$. The singular values are obtained by taking the square roots of its eigenvalues.

Problem 23. Stack the singular values of $\mathbf{B}^T\mathbf{B}$ diagonally to obtain a matrix $\mathbf{\Sigma}$.

Problem 24. Obtain the matrix $\mathbf{BV}$. Verify if the columns of this matrix are orthogonal.

Problem 25. Extend the columns of $\mathbf{BV}$ if necessary, to obtain an orthogonal matrix $\mathbf{U}$.

Problem 26. Find $\mathbf{U\Sigma V}^T$. Comment.

2.5 Quadratic Forms

Problem 27. $\theta = \mathbf{x}^T\mathbf{Cx}$ is known as the *Quadratic Form* for $\mathbf{C}$. θ is defined for a *Symmetric Matrix*. A matrix for which the quadratic form is always positive is known as a *positive definite* matrix. Is $\mathbf{C}$ positive definite?

Problem 28. Find out the relation between positive definiteness and the eigenvalues of a symmetric matrix.

Problem 29. Find the minimum and maximum values of $\theta = \mathbf{x}^T\mathbf{Cx}$, if $\|\mathbf{x}\| = 1$.