Matrix Analysis through Python

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Contents

1 Least Squares

1.1 Problem

Problem 1. Sketch the vectors

$$\mathbf{a}_1 = (1,1,1)^T, \mathbf{a}_2 = (0,1,2)^T, \mathbf{b} = (6,0,0)^T \tag{1.1}$$

in the 3-D plane.

Problem 2. Find x_1, x_2 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \tag{2.1}$$

geometrically.

Problem 3. Solve the matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.1}$$

where $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2]$ using row reduction. Comment.

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1.2 Solution using Python

Problem 4. Run the following Python code and comment on the output for different values of $\mathbf{x}$

```
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
from numpy import linalg as LA

#Creating matrix
A=np.matrix('1 0; 1 1; 1 2')
#Creating vector
b=np.matrix('6; 0; 0')
#P = inv((A'A)A'
P=np.dot(inv(np.dot(np.transpose(A
    ), A)), np.transpose(A))
#x_ls=Pb
x_ls=np.dot(P, b)
x=np.matrix('5; -5')
#||b-Ax_ls||
exact_ls_metric=(LA.norm(b-np.dot(
    A, x_ls)))**2
#||b-Ax||
random_ls_metric=(LA.norm(b-np.dot
    (A, x)))**2
print(x_ls)
print(x)
print(exact_ls_metric)
print(random_ls_metric)
```

Problem 5. Compare the results obtained by typing the following code with the results in the previous problem.

```
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
from numpy import linalg as LA

A=np.matrix('1 0; 1 1; 1 2')
b=np.matrix('6; 0; 0')
```

```
#SVD, A=USV
U, s, V=LA.svd(A)
#Find size of A
mn=A.shape
#Creating the singular matrix
S = np.zeros(mn)
Sinv = S.T
S[:2,:2] = np.diag(s)
#Verifying the SVD, A=USV
print(U.dot(S).dot(V))
#Inverting s
sinv = 1./s
#Inverse transpose of S
Sinv[:2,:2] = np.diag(sinv)
print(Sinv)
#Moore-Penrose Pseudoinverse
Aplus = V.T.dot(Sinv).dot(U.T)
#Least squares solution
x_ls = Aplus.dot(b)
#
print(x_ls)
```

Problem 6. Type the following code in Python and run. Comment.

```
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
from numpy import linalg as LA

#Creating matrices
A=np.matrix('1 0; 1 1; 1 2')
b=np.matrix('6; 0; 0')
#Eigenvalue decompostion of A'A
Dv, Pv=LA.eig(A.T.dot(A))
#Eigenvalue decompostion of AA'
Du, Pu=LA.eig(A.dot(A.T))
#Singular values of A
Stemp=np.sqrt(Dv)
#QR Decomposition to get U and V
V, Rv=LA.qr(Pv)
U, Ru=LA.qr(Pu)
#SVD, A=USV
U_1, s, V_1=LA.svd(A)
print(V)
print(V_1)
print(U)
print(U_1)
print(s)
print(Stemp)
```

Let

$$g(\mathbf{x}) = \|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2 \tag{6.1}$$

Problem 7. Using calculus, minimize $g(\mathbf{x})$.

Problem 8. Find $(A^T A)^{-1} A^T b$

2 Matrix Analysis

Verify your results through Python, wherever possible.

2.1 Eigenvalues and Eigenvectors

For any square matrix $\mathbf{G}$, if

$$\mathbf{G}\mathbf{x} = \lambda\mathbf{x}, \tag{8.1}$$

λ is known as the *eigenvalue* and $\mathbf{x}$ is the corresponding *eigenvector*.

Let

$$\mathbf{G} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \tag{8.2}$$

Problem 9. Show that the eigenvalues of $\mathbf{G}$ are obtained by solving the equation

$$f(\lambda) = |\lambda\mathbf{I} - G| = 0 \tag{9.1}$$

Note that (9.1) is known as the *characteristic equation*. $f(\lambda)$ is known as the characteristic polynomial.

Problem 10. Obtain the eigenvalues and eigenvectors of $\mathbf{G}$.

Problem 11. Find $f(\mathbf{G})$. This is known as the *Cayley-Hamilton Theorem*.

Problem 12. Stack the eigenvalues of $\mathbf{G}$ in a diagonal matrix $\mathbf{\Lambda}$ and the corresponding eigenvectors in a matrix $\mathbf{F}$. Find $\mathbf{F}\mathbf{\Lambda}\mathbf{F}^{-1}$. This is known as *Eigenvalue Decomposition*

2.2 Symmetric Matrices

Let

$$\mathbf{C} = \begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix} \tag{12.1}$$

Note that $\mathbf{C} = \mathbf{C}^T$. Such matrices are known as *symmetric matrices*.

Problem 13. Find $\mathbf{P}$ such that $\mathbf{C} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

Problem 14. Find $\mathbf{PP}^T$ and $\mathbf{P}^T\mathbf{P}$. $\mathbf{P}$ is known as an *orthogonal matrix.*

Let

$$\mathbf{B} = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix} \tag{14.1}$$

Problem 15. Find $\mathbf{B}^T\mathbf{B}$ and $\mathbf{BB}^T$

Note that $\mathbf{C} = \frac{1}{9}\left(\mathbf{BB}^T\right)$.

Problem 16. Obtain the eigenvalues and eigenvectors of $\mathbf{B}^T\mathbf{B}$

Problem 17. Verify eigenvalue decomposition and Cayley-Hamilton theorem for $\mathbf{B}^T\mathbf{B}$.

2.3 Orthogonality

Let $\mathbf{v}_1, \mathbf{v}_2$ be the columns of $\mathbf{C}$.

Problem 18. Obtain $\mathbf{u}_1, \mathbf{u}_2$ from $\mathbf{v}_1, \mathbf{v}_2$ through the following equations.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||} \tag{18.1}$$

$$\hat{\mathbf{u}}_2 = \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\,\mathbf{u}_1 \tag{18.2}$$

$$\mathbf{u}_2 = \frac{\hat{\mathbf{u}}_2}{||\hat{\mathbf{u}}_2||} \tag{18.3}$$

This procedure is known as Gram-Schmidt orthogonalization.

Problem 19. Stack the vectors $\mathbf{u}_1, \mathbf{u}_2$ in columns to obtain the matrix $\mathbf{Q}$. Show that $\mathbf{Q}$ is orthogonal.

Problem 20. From the Gram=Schmidt process, show that $\mathbf{C} = \mathbf{QR}$, where $\mathbf{R}$ is an upper triangular matrix. This is known as the $\mathbf{Q} - \mathbf{R}$ decomposition.

2.4 Singular Value Decomposition

Problem 21. Find an orthonormal basis for $\mathbf{B}^T\mathbf{B}$ comprising of the eigenvectors. Stack these orthonormal eigenvectors in a matrix $\mathbf{V}$. This is known as *Orthogonal Diagonalization*.

Problem 22. Find the singular values of $\mathbf{B}^T\mathbf{B}$. The singular values are obtained by taking the square roots of its eigenvalues.

Problem 23. Stack the singular values of $\mathbf{B}^T\mathbf{B}$ diagonally to obtain a matrix $\mathbf{\Sigma}$.

Problem 24. Obtain the matrix $\mathbf{BV}$. Verify if the columns of this matrix are orthogonal.

Problem 25. Extend the columns of $\mathbf{BV}$ if necessary, to obtain an orthogonal matrix $\mathbf{U}$.

Problem 26. Find $\mathbf{U\Sigma V}^T$. Comment.

2.5 Quadratic Forms

Problem 27. $\theta = \mathbf{x}^T\mathbf{Cx}$ is known as the *Quadratic Form* for $\mathbf{C}$. θ is defined for a *Symmetric Matrix*. A matrix for which the quadratic form is always positive is known as a *positive definite* matrix. Is $\mathbf{C}$ positive definite?

Problem 28. Find out the relation between positive definiteness and the eigenvalues of a symmetric matrix.

Problem 29. Find the minimum and maximum values of $\theta = \mathbf{x}^T\mathbf{Cx}$, if $||\mathbf{x}|| = 1$.

3 Application in Research

Problem 30. Let

$$r = \sum_{j=1}^{2} h_j c_j \tag{30.1}$$

Express the above as a matrix equation. Note that r is a scalar.

Problem 31. Let

$$r_i = \sum_{j=1}^{2} h_{ij} c_j, \quad i = 1, 2. \tag{31.1}$$

Express the above as the matrix equation

$$\mathbf{r} = \mathbf{Hc} \tag{31.2}$$

List the entries of each matrix/vector in (31.2).

Problem 32. If

$$r_i = \sum_{j=1}^{N} h_{ij} c_j, \quad i = 1, 2 \ldots M, \tag{32.1}$$

what is the dimension of the matrix $\mathbf{H}$ in the matrix equation?

Problem 33. Let

$$\mathbf{r}^t = \mathbf{h}^t\mathbf{C} \tag{33.1}$$

where $\mathbf{r}$ is $L \times 1$ vector and C is an $N \times L$ matrix. Find the least squares estimate for $\mathbf{h}$. What is the size of $\mathbf{h}$?

Problem 34. Now consider the matrix equation

$$\mathbf{R} = \mathbf{HC} \tag{34.1}$$

where $\mathbf{R}$ is $M \times L$, $\mathbf{H}$ is $M \times N$ and $\mathbf{C}$ is $N \times L$. Find the least squares estimate of $\mathbf{H}$.

Problem 35. Let

$$D = x_1^2 - x_2^2 \tag{35.1}$$

D can be expressed in quadratic form as $D = \mathbf{x}^t Q \mathbf{x}$, where $\mathbf{x} = (x_1, x_2)^t$. Find Q.

Problem 36. Find the determinant and eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \tag{36.1}$$

Problem 37. Find the determinant and eigenvalues of $\mathbf{A} \otimes \mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix. Comment.

Problem 38. Find the eigenvalues of $I - kA$, without explicitly calculating them. k is a constant.

Consider the matrix

$$\mathbf{S} = \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \tag{38.1}$$

where $*$ represents the conjugate of a scalar and conjugate transpose of a vector.

Problem 39. Find SS^*. Comment.

Problem 40. Express

$$\begin{aligned} r_1 &= h_1 s_1 + h_2 s_2 \\ r_2 &= -h_1 s_2^* + h_2 s_1^* \end{aligned} \tag{40.1}$$

as a matrix equation.

Problem 41. Solve for s_1 and s_2 in (40.1) using matrices.

The problems in this chapter were framed using [1] and [2]. The primary reference for this manual is [3].

References

[1] P. Garg, R. K. Mallik, and H. M. Gupta, "Performance Analysis of Space-Time Coding with Imperfect Channel Estimation," *IEEE Trans. Wireless Commun.*, vol. 4, no. 1, pp. 257–265, January 2005.

[2] S. M. Alamouti, "A Simple Transmitter Diversity Scheme for Wireless Communications," *IEEE J. Sel. Areas Commun.*, vol. 16, p. 14511458, October 1998.

[3] D. C. Lay, *Linear Algebra and its Applications*. Addison-Wesley, 1993.