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Applied Probability

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The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right\} \quad (1)$$

where μ is the mean vector, $\Sigma = E\left[(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T\right]$ is the covariance matrix and $|\Sigma|$ is the determinant of Σ.

Problem 1. Show that

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right\}\right] \quad (2)$$

where

$$\mu = \begin{pmatrix}\mu_x \\ \mu_y\end{pmatrix}, \Sigma = \begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix} \quad (3)$$

Problem 2. If

$$\mathbf{y}|0 = \begin{pmatrix}\sqrt{A}+n_1 \\ n_2\end{pmatrix}, \quad (4)$$

and

$$\mathbf{y}|1 = \begin{pmatrix}n_1 \\ \sqrt{A}+n_2\end{pmatrix}, \quad (5)$$

use the MAP criterion to reach a decision.

Problem 3. Derive and plot the probability of error. Verify through simulation.

Problem 4. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \quad (6)$$

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where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix}A \\ 0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0 \\ A\end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix}-A \\ 0\end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix}0 \\ -A\end{pmatrix}, \quad (7)$$

$$E[\mathbf{n}] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \quad (8)$$

1) Show that the MAP decision for detecting $\mathbf{s}_0$ results in
$$|r|_2 < r_1 \quad (9)$$
2) Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as
$$\rho = \frac{E\left[(X-\mu_x)\left(Y-\mu_y\right)\right]}{\sigma_x\sigma_y} \quad (10)$$
X and Y are said to be uncorrelated if $\rho = 0$
3) Show that if X and Y are uncorrelated Verify this numerically.
4) Show that X and Y are independent, i.e. $p_{XY}(x,y) = p_X(x)p_Y(y)$.
5) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.
6) Show that $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < A, Y < A)$.
7) Find $\Pr(X < A, Y < A)$.
8) Verify the above through simulation.

Problem 5. Show that

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta \quad (11)$$

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\theta-\phi)}\, d\theta \quad (12)$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{m_1\cos\theta + m_2\sin\theta}\, d\theta = I_0\left(\sqrt{m_1^2+m_2^2}\right) \quad (13)$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos n\theta\, d\theta \quad (14)$$

Problem 6. Let

$$\mathbf{r}|0 = \sqrt{E_b}\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\\ 0\\ 0\end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b}\begin{pmatrix}0\\ 0\\ \cos\phi_1\\ \sin\phi_1\end{pmatrix} + \mathbf{n}_1 \tag{15}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

1) Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$,, find the pdf $p(\mathbf{r}|0, \phi_0)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.
2) If ϕ_0 is uniformly distributed between 0 and 2π, find $p(\mathbf{r}|0)$. Note that this expression will no longer contain ϕ_0.
3) Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2 + r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2 + r_4^2}\right) \tag{16}$$

   where k is a constant.
4) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.
5) Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2|0\right) \tag{17}$$

6) Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0}e^{-\frac{y}{N_0}}, y > 0 \tag{18}$$

7) Find

$$g(r_1, r_2) = \Pr\left(r_1^2 + r_2^2 < X|0, r_1, r_2\right). \tag{19}$$

8) Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}}e^{-\frac{\mu^2}{2\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.
9) Now show that

$$E\left[g(r_1, r_2)\right] = \frac{1}{2}e^{-\frac{E_b}{2N_0}}. \tag{20}$$

Problem 7. Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R}\cos\Theta \tag{21}$$

$$V = \sqrt{R}\sin\Theta \tag{22}$$

1) Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det\begin{pmatrix}\frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta}\\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta}\end{pmatrix} \tag{23}$$

2) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}(r, \theta) = p_{U,V}(u, v)\, J|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{24}$$

   Show that

$$p_R(r) = \begin{cases}\frac{1}{k}e^{-\frac{r}{k}} & r > 0,\\ 0 & r < 0,\end{cases} \tag{25}$$

   assuming that Θ is uniformly distributed between 0 to 2π.
3) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k}e^{-\frac{|y|}{k}} \tag{26}$$

4) Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{27}$$

   where ϕ is a constant.
5) Find $\Pr(Y > Z)$.
6) If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{28}$$

   is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k}I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{29}$$

7) Show that

$$I_0(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{4^n(n!)^2} \tag{30}$$

8) If

$$p_Z(z) = \begin{cases}\frac{1}{k}e^{-\frac{z}{k}} & z \geq 0\\ 0 & z < 0\end{cases} \tag{31}$$

   find $\Pr(R < Z)$.

Problem 8. Consider a system where $\mathbf{s}_i = \begin{pmatrix}\cos\left(\frac{2\pi i}{M}\right)\\ \cos\left(\frac{2\pi i}{M}\right)\end{pmatrix}, i = 0, 1, \ldots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix}r_1\\ r_2\end{pmatrix} = \begin{pmatrix}\sqrt{E_s} + n_1\\ n_2\end{pmatrix} \tag{32}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

1) Substituting

$$r_1 = R\cos\theta \tag{33}$$
$$r_2 = R\sin\theta \tag{34}$$

show that the joint pdf of R, θ is

$$p(R,\theta) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{35}$$

2) Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\, e^{-(V-\alpha)^2}\, dV = 0 \tag{36}$$
$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\, dV = \sqrt{\pi} \tag{37}$$

3) Using the above, evaluate

$$\int_0^\infty V\exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \tag{38}$$

for large values of γ.

4) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\, d\theta \tag{39}$$

5) Find $P_{e|\mathbf{s}_0}$.

Problem 9. The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{40}$$

where X is a random variable and $E[\cdot]$ is the expectation.

1) Let $Y \sim \mathcal{N}(0,1)$. Define

$$Q(x) = \Pr(Y > x),\, x > 0 \tag{41}$$

Show that

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{42}$$

2) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

3) Let

$$P_e = \Pr\left(\Re\left\{h^* y\right\} < 0\right), \text{ where } y = \left(\sqrt{E_s}h + n\right), \tag{43}$$

Show that

$$P_e = \int_0^\infty Q\left(\sqrt{2x}\right) p_A(x)\, dx \tag{44}$$

where $A = \frac{E_s|h|^2}{N_0}$.

4) Show that

$$P_e = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{45}$$

5) compute $M_A(s)$.
6) Find P_e.
7) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.