Matrix Analysis through Octave

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Contents

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1 Least Squares

1.1 Problem

Problem 1.1. *Sketch the vectors*

$$\mathbf{a}_1 = (1, 1, 1)^T, \mathbf{a}_2 = (0, 1, 2)^T, \mathbf{b} = (6, 0, 0)^T \tag{1.1}$$

in the 3-D plane.

Problem 1.2. *Find x_1, x_2 such that*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \tag{1.2}$$

geometrically.

Problem 1.3. *Solve the matrix equation*

$$\mathbf{Ax} = \mathbf{b} \tag{1.3}$$

where $\mathbf{A} = [\mathbf{a}_1\, \mathbf{a}_2]$ using row reduction. Comment.

1.2 Solution using Octave

Problem 1.4. *Type the following program in octave and comment on the output for different values of* $\mathbf{x}$

```
%Code written by GVV Sharma March 30, 2016
%Released under GNU GPL.  Free to use for anything.


%This program compares the norm defined for the least-squares solution
%for the correct solution vs other data points.
%You will find that the metric is the smallest for the correct value.


clear;
close;
```

```
A = [1 0; 1 1; 1 2]; %The input matrix
b = [6;0;0]; %The output vector

P = inv(A'*A)*A';%pseudoinverse

x_ls = P*b; %The least squares solution

x = [5;-5]; %Any random input

exact_ls_metric = norm(b-A*x_ls)ˆ2 %The metric for actual soltuion
random_ls_metric = norm(b-A*x)ˆ2 %metric for a random value of x
```

Problem 1.5. *Type the following code in Octave and observe the output.*

```
%Code written by GVV Sharma March 31, 2016
%Released under GNU GPL.  Free to use for anything.



%This program plots the least squares metric for a range of
%vectors x in the mesh with vertices (-10,-10),(-10,10),(10,-10)
%%and (10,10)

%The result is a 3-D mesh.  The theoretical minimum is (5,-3)
%Values obtained through the following program are close to the
%theoreticl solution
```

```
clear;

close;


A = [1 0; 1 1; 1 2]; %The input matrix

b = [6;0;0]; %The output vector



x1 = linspace(-10,10,50); %generating points in x-axis

x2 = linspace(-10,10,50);  %generating points in y-axis


[xx, yy] = meshgrid(x1,x2);


ffun = @(x,y) norm(b-A*[x;y])^2;


f = arrayfun(ffun,xx,yy);


mesh(xx,yy,f)


[M I] = min(f(:)); %vectorize the 50 x 50 matrix f, find min

%M = min value , I is the index of the f_min


[I_r I_c] = ind2sub(size(f),I); %Get the row, col index of f_min
```

```
%The least square solution
xx(I_r,I_c)
yy(I_r,I_c)
%The minimum value of metric
M
```

Problem 1.6. *Compare the results obtained by typing the following code with the results in the previous problem.*

```
%Code written by GVV Sharma March 31, 2016
%Released under GNU GPL.  Free to use for anything.


%This program finds the theoretical least squares solution using
%SVD

clear;
close;

A = [1 0; 1 1; 1 2]; %The input matrix
b = [6;0;0]; %The output vector


[U S V] = svd(A); % Computing the SVD of A
```

```
temp_S = 1./diag(S); %inverting the diagonal values of S

Splus = [diag(temp_S) zeros(2,1)]; %inverse transpose of S

Aplus = V*Splus*U'; %The Moore-Penrose pseudo-inverse

Aplus*b %least squares solution.
```

Problem 1.7. *Type the following code in Octave and run. Comment.*

```
%Code written by GVV Sharma March 31, 2016
%Released under GNU GPL.  Free to use for anything.


%This program finds the SVD for the matrix A
%Involves eigenvalue decomposition as well as
%QR factorization (Gram-Schmidt Orthogonalization)

%Note that the columns of U and V are interchanged
%when compared with the U and V matrices obtained
%using the builtin SVD command.


clear;
close;
```

```
A = [1 0; 1 1; 1 2]; %The input matrix

b = [6;0;0]; %The output vector



[Pv,Dv] = eig(A'*A);%Eigenvalue decomposition of A'*A

[Pu,Du] = eig(A*A');%Eigenvalue decomposition of A*A'



Stemp = sqrt(Dv); %singular values of A

[V,Rv] = qr(Pv); %V

[U,Ru] = qr(Pu);  %U
```

Let

$$g(\mathbf{x}) = ||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 \tag{1.4}$$

Problem 1.8. *Using calculus, minimize* $g(\mathbf{x})$*.*

Problem 1.9. *Find* $(A^T A)^{-1} A^T b$

2 Matrix Analysis

Verify your results through Octave, wherever possible.

2.1 Eigenvalues and Eigenvectors

For any square matrix $\mathbf{G}$, if

$$\mathbf{G}\mathbf{x} = \lambda\mathbf{x}, \tag{2.1}$$

λ is known as the *eigenvalue* and $\mathbf{x}$ is the corresponding *eigenvector*.

Let

$$\mathbf{G} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \tag{2.2}$$

Problem 2.1. *Show that the eigenvalues of* $\mathbf{G}$ *are obtained by solving the equation*

$$f(\lambda) = |\lambda\mathbf{I} - G| = 0 \tag{2.3}$$

Note that (2.3) is known as the *characteristic equation*. $f(\lambda)$ is known as the characteristic polynomial.

Problem 2.2. *Obtain the eigenvalues and eigenvectors of* $\mathbf{G}$.

Problem 2.3. *Find* $f(\mathbf{G})$. *This is known as the* Cayley-Hamilton Theorem.

Problem 2.4. *Stack the eigenvalues of* $\mathbf{G}$ *in a diagonal matrix* $\mathbf{\Lambda}$ *and the corresponding eigenvectors in a matrix* $\mathbf{F}$. *Find* $\mathbf{F}\mathbf{\Lambda}\mathbf{F}^{-1}$. *This is known as* Eigenvalue Decomposition

2.2 Symmetric Matrices

Let

$$\mathbf{C} = \begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix} \tag{2.4}$$

Note that $\mathbf{C} = \mathbf{C}^T$. Such matrices are known as *symmetric matrices*.

Problem 2.5. *Find* $\mathbf{P}$ *such that* $\mathbf{C} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, *where* $\mathbf{D}$ *is a diagonal matrix.*

Problem 2.6. *Find* $\mathbf{P}\mathbf{P}^T$ *and* $\mathbf{P}^T\mathbf{P}$. $\mathbf{P}$ *is known as an* orthogonal matrix.

Let

$$\mathbf{B} = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix} \tag{2.5}$$

Problem 2.7. *Find* $\mathbf{B}^T\mathbf{B}$ *and* $\mathbf{B}\mathbf{B}^T$

Note that $\mathbf{C} = \frac{1}{9}\left(\mathbf{B}\mathbf{B}^T\right)$.

Problem 2.8. *Obtain the eigenvalues and eigenvectors of* $\mathbf{B}^T\mathbf{B}$

Problem 2.9. *Verify eigenvalue decomposition and Cayley-Hamilton theorem for* $\mathbf{B}^T\mathbf{B}$.

2.3 Orthogonality

Let $\mathbf{v}_1, \mathbf{v}_2$ be the columns of $\mathbf{C}$.

Problem 2.10. *Obtain* $\mathbf{u}_1, \mathbf{u}_2$ *from* $\mathbf{v}_1, \mathbf{v}_2$ *through the following equations.*

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \tag{2.6}$$

$$\hat{\mathbf{u}}_2 = \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\,\mathbf{u}_1 \tag{2.7}$$

$$\mathbf{u}_2 = \frac{\hat{\mathbf{u}}_2}{\|\hat{\mathbf{u}}_2\|} \tag{2.8}$$

This procedure is known as Gram-Schmidt orthogonalization.

Problem 2.11. *Stack the vectors* $\mathbf{u}_1, \mathbf{u}_2$ *in columns to obtain the matrix* $\mathbf{Q}$. *Show that* $\mathbf{Q}$ *is orthogonal.*

Problem 2.12. *From the Gram=Schmidt process, show that* $\mathbf{C} = \mathbf{QR}$, *where* $\mathbf{R}$ *is an upper triangular matrix. This is known as the* $\mathbf{Q} - \mathbf{R}$ *decomposition.*

2.4 Singular Value Decomposition

Problem 2.13. *Find an orthonormal basis for* $\mathbf{B}^T\mathbf{B}$ *comprising of the eigenvectors. Stack these orthonormal eigenvectors in a matrix* $\mathbf{V}$. *This is known as* Orthogonal Diagonalization.

Problem 2.14. *Find the singular values of* $\mathbf{B}^T\mathbf{B}$. *The singular values are obtained by taking the square roots of its eigenvalues.*

Problem 2.15. *Stack the singular values of* $\mathbf{B}^T\mathbf{B}$ *diagonally to obtain a matrix* $\mathbf{\Sigma}$.

Problem 2.16. *Obtain the matrix* $\mathbf{BV}$. *Verify if the columns of this matrix are orthogonal.*

Problem 2.17. *Extend the columns of* $\mathbf{BV}$ *if necessary, to obtain an orthogonal matrix* $\mathbf{U}$.

Problem 2.18. *Find* $\mathbf{U\Sigma V}^T$. *Comment.*

2.5 Quadratic Forms

Problem 2.19. *Type the following in Octave and interpret the output.* $\theta = \mathbf{x}^T\mathbf{Cx}$ *is known as the* Quadratic Form *for* $\mathbf{C}$. θ *is defined for a* Symmetric Matrix.

```
%Code written by GVV Sharma April 10, 2016
%Released under GNU GPL.  Free to use for anything.


%This program plots the quadratic form for a range of
%vectors x in the mesh with vertices (-10,-10),(-10,10),(10,-10)
%%and (10,10)


%The result is a 3-D mesh.
%The quadratic form in terms of the eigenvalues of the
%symmetric matrix is explored through this program.



clear;
close;
```

```
C = [37 9; 9 13];
[P lambda] = eig(C);

x1 = linspace(-10,10,50); %generating points in x-axis
x2 = linspace(-10,10,50);  %generating points in y-axis

[xx, yy] = meshgrid(x1,x2);

ffun = @(x,y) [x y]*C*[x;y];

f = arrayfun(ffun,xx,yy);

mesh(xx,yy,f)

[M I] = min(f(:)); %vectorize the 50 x 50 matrix f, find min
%M = min value , I is the index of the f_min

[I_r I_c] = ind2sub(size(f),I); %Get the row, col index of f_min

%The minimum value of the quadratic form
M
%Verifying the eigenvalue relation
x_hat = [xx(I_r,I_c);  yy(I_r,I_c)]
x_hat'*C*x_hat
```

```
z = P*[xx(I_r,I_c);  yy(I_r,I_c)]

z'*lambda*z
```

Problem 2.20. *A matrix for which the quadratic form is always positive is known as a* positive definite *matrix. Is* **C** *is positive definite?*

Problem 2.21. *Find out the relation between positive definiteness and the eigenvalues of a symmetric matrix.*

Problem 2.22. *Find the minimum and maximum values of* $\theta = \mathbf{x}^T\mathbf{C}\mathbf{x}$*, if* $||\mathbf{x}|| = 1$.

3 Application in Research

Problem 3.1. *Let*

$$r = \sum_{j=1}^{2} h_j c_j \tag{3.1}$$

Express the above as a matrix equation. Note that r is a scalar.

Problem 3.2. *Let*

$$r_i = \sum_{j=1}^{2} h_{ij} c_j, \quad i = 1, 2. \tag{3.2}$$

Express the above as the matrix equation

$$\mathbf{r} = \mathbf{H}\mathbf{c} \tag{3.3}$$

List the entries of each matrix/vector in (3.3).

Problem 3.3. *If*

$$r_i = \sum_{j=1}^{N} h_{ij} c_j, \quad i = 1, 2 \dots M, \tag{3.4}$$

what is the dimension of the matrix $\mathbf{H}$ *in the matrix equation?*

Problem 3.4. *Let*

$$\mathbf{r}^t = \mathbf{h}^t \mathbf{C} \tag{3.5}$$

where $\mathbf{r}$ *is $L \times 1$ vector and C is an $N \times L$ matrix. Find the least squares estimate for* $\mathbf{h}$. *What is the size of* $\mathbf{h}$*?*

Problem 3.5. *Now consider the matrix equation*

$$\mathbf{R} = \mathbf{H}\mathbf{C} \tag{3.6}$$

where $\mathbf{R}$ *is $M \times L$,* $\mathbf{H}$ *is $M \times N$ and* $\mathbf{C}$ *is $N \times L$. Find the least squares estimate of* $\mathbf{H}$.

Problem 3.6. *Let*

$$D = x_1^2 - x_2^2 \tag{3.7}$$

D can be expressed in quadratic form as $D = \mathbf{x}^t Q \mathbf{x}$, where $\mathbf{x} = (x_1, x_2)^t$. Find Q.

Problem 3.7. *Find the determinant and eigenvalues of*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \tag{3.8}$$

Problem 3.8. *Find the determinant and eigenvalues of* $\mathbf{A}\otimes\mathbf{I}$*, where* $\mathbf{I}$ *is the* 2×2 *identity matrix. Comment.*

Problem 3.9. *Find the eigenvalues of* $I - kA$*, without explicitly calculating them.* k *is a constant.*

Consider the matrix

$$\mathbf{S} = \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \tag{3.9}$$

where $*$ represents the conjugate of a scalar and conjugate transpose of a vector.

Problem 3.10. *Find* SS^**. Comment.*

Problem 3.11. *Express*

$$\begin{aligned} r_1 &= h_1 s_1 + h_2 s_2 \\ r_2 &= -h_1 s_2^* + h_2 s_1^* \end{aligned} \tag{3.10}$$

as a matrix equation.

Problem 3.12. *Solve for* s_1 *and* s_2 *in* (3.10) *using matrices.*

The problems in this chapter were framed using [1] and [2]. The primary reference for this manual is [3].

References

[1] P. Garg, R. K. Mallik, and H. M. Gupta, "Performance Analysis of Space-Time Coding with Imperfect Channel Estimation," *IEEE Trans. Wireless Commun.*, vol. 4, no. 1, pp. 257–265, January 2005.

[2] S. M. Alamouti, "A Simple Transmitter Diversity Scheme for Wireless Communications," *IEEE J. Sel. Areas Commun.*, vol. 16, p. 14511458, October 1998.

[3] D. C. Lay, *Linear Algebra and its Applications*. Addison-Wesley, 1993.