Simplex Method

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***Abstract*—This manual explains the Simplex Method for solving Linear Programming problems through examples.**

1 Iterations and Pivoting

Problem 1.1. Maximize

$$f = 6x_1 + 5x_2 \tag{1.1}$$

with constraints

$$x_1 + x_2 \le 5$$
$$3x_1 + 2x_2 \le 12$$
$$\text{where } x_1, x_2 \ge 0$$

Solution: Introduce two dummy variables x_3, x_4 to convert inequalities to equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_4 &= 12 \\ \text{where } x_3, x_4 &\ge 0 \end{aligned} \tag{1.2}$$

1. From (1.2), we obtain

$$x_3 = 5 - x_1 - x_2 \tag{1.3}$$
$$x_4 = 12 - 3x_1 - 2x_2 \tag{1.4}$$

Setting $x_2 = 0$ in (1.4),

$$\begin{aligned} x_3 > 0 &\Rightarrow 5 - x_1 > 0 \Rightarrow x_1 < 5 \\ x_4 > 0 &\Rightarrow 12 - 3x_1 > 0 \Rightarrow x_1 < 4 \end{aligned} \tag{1.5}$$

and

$$f_1 = 6x_1 + 5x_2 \tag{1.6}$$

2. (Pivoting): (1.4) results in a lower bound for x_1. Rearranging (1.4),

$$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4 \tag{1.7}$$
$$\begin{aligned} x_3 &= 5 - x_1 - x_2 \\ &= 5 - \left(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4\right) - x_2 \\ &= 1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 \end{aligned} \tag{1.8}$$

and substituting $x_4 = 0$ results in

$$x_1 > 0 \implies x_2 < 6 \tag{1.9}$$
$$x_3 > 0 \implies x_2 < 3 \tag{1.10}$$

and

$$f_2 = 6\left(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4\right) + 5x_2 \tag{1.11}$$
$$= 24 + x_2 - 2x_4 \tag{1.12}$$

3. The lower bound $x_2 < 3$ results from (1.8). Choosing (1.8) for pivoting for x_2 and rearranging,

$$x_2 = 1 + x_4 - 3x_3, \tag{1.13}$$
$$x_1 = 4 - \frac{2}{3}(1 + x_4 - 3x_3) - \frac{1}{3}x_4 \tag{1.14}$$
$$= \frac{10}{3} - x_4 + 2x_3 \tag{1.15}$$

and

$$f_3 = 24 + x_2 - 2x_4 \tag{1.16}$$
$$= 24 + (1 + x_4 - 3x_3) - 2x_4 \tag{1.17}$$
$$= 25 - x_4 - 3x_3 \tag{1.18}$$

Since $x_3, x_4 \ge 0$, the maximum value of $f_3 = 25$ and this is the desired answer. The iteration ends when the coefficients of the variables in f_i, where i is the ith iteration are all negative.

5) Complete iteration three, pivoting x_2, and find the maximum value of the expression. Solve

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for corresponding x_1 and x_2

Problem 1.2. Maximise $5x_1 + 3x_2$ w.r.t the constraints

$$\begin{aligned} x_1 + x_2 &\le 2 \\ 5x_1 + 2x_2 &\le 10 \\ 3x_1 + 8x_2 &\le 12 \\ \text{where } x_1, x_2 &\ge 0 \end{aligned}$$

2 Tabular Method

Problem 2.1. Maximize

$$f = 6x_1 + 8x_2 \tag{2.1}$$

with constraints:

$$x_1 + x_2 \le 10 \tag{2.2}$$
$$2x_1 + 3x_2 \le 25 \tag{2.3}$$
$$x_1 + 5x_2 \le 35 \tag{2.4}$$

Solution: Introducing dummy variables x_3, x_4, x_5

$$x_1 + x_2 + x_3 = 10 \tag{2.5}$$
$$2x_1 + 3x_2 + x_4 = 25 \tag{2.6}$$
$$x_1 + 5x_2 + x_5 = 35 \tag{2.7}$$

The objective function in (2.1) becomes

$$f = 6x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5 \tag{2.8}$$

1. Set up a table as below. Note that the numbers in the first row are the coefficents of x_1, x_2 in (2.1).

		6	8	0	0	0	
		x_1	x_2	x_3	x_4	x_5	RHS
0	x_3	1	1	1	0	0	10
0	x_4	2	3	0	1	0	25
0	x_5	1	5	0	0	1	35

TABLE 2.1.1

2. Define, c_j = coefficient of x_j in (2.8), where $j = 1, 2, 3, 4, 5$.

$$z_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}.\begin{pmatrix}1\\2\\1\end{pmatrix} \tag{2.9}$$

$$z_2 = \begin{pmatrix}0\\0\\0\end{pmatrix}.\begin{pmatrix}1\\3\\5\end{pmatrix} \tag{2.10}$$

$$\vdots$$

$$z_{RHS} = \begin{pmatrix}0\\0\\0\end{pmatrix}.\begin{pmatrix}10\\25\\35\end{pmatrix} \tag{2.11}$$

where the scalar products of the column vectors are being computed.

		6	8	0	0	0	
		x_1	x_2	x_3	x_4	x_5	RHS
0	x_3	1	1	1	0	0	10
0	x_4	2	3	0	1	0	25
0	x_5	1	5	0	0	1	35
	$c_j - z_j$	6	**8**↑	0	0	0	z_{RHS} =0

TABLE 2.1.2

3. For $j = 2, c_j - z_j = 8 - 0 = 8$ which is the largest among all j. So x_2 is going to enter into next iteration. Which variable is going to leave? Make a new column θ as in Table 2.1.3 where

$$\theta = \frac{\text{RHS value}}{\text{corresponding values in column of } x_2} \tag{2.12}$$

		6	8	0	0	0		
		x_1	x_2	x_3	x_4	x_5	RHS	θ
0	x_3	1	1	1	0	0	10	10
0	x_4	2	3	0	1	0	25	$\frac{25}{3}$
0	x_5	1	**5**	0	0	1	35	**7**⟶
	$c_j - z_j$	6	**8**↑	0	0	0	z_{RHS} =0	

TABLE 2.1.3

4. From Table 2.1.3, it is clear that 7 is the smallest value of θ and corresponds to the row with x_5. So x_5 is going to leave the iteration. 5 in Table 2.1.3 corresponds to the x_2 column (max $c_j - z_j$) and x_5 row (min θ) is called the **pivot**. The pivot element should be made 1 in the next iteration

$$Row_{x_5} = \frac{Row_{x_5}}{5} \tag{2.13}$$

5. As x_5 is leaving, this becomes row corresponding to x_2. All the other elements of column containing pivot should be made 0 by row operations. See Table 2.1.4.

$$R_{x_3} = R_{x_3} - R_{x_2} \tag{2.14}$$
$$R_{x_4} = R_{x_4} - 3R_{x_2} \tag{2.15}$$
$$\tag{2.16}$$

		6	8	0	0	0		
		x_1	x_2	x_3	x_4	x_5	RHS	θ
0	x_3	1	1	1	0	0	10	10
0	x_4	2	3	0	1	0	25	$\frac{25}{3}$
0	x_5	1	**5**	0	0	1	35	**7**$\longrightarrow$
	$c_j - z_j$	6	**8**$\uparrow$	0	0	0	z_{RHS} =0	
0	x_3	4/5	0	1	0	-1/5	3	15/4
0	x_4	**2/5**	0	0	1	-3/5	4	**20/7** $\longrightarrow$
8	x_2	1/5	1	0	0	1/5	7	35
	$c_j - z_j$	**22/5**$\uparrow$	0	0	0	-8/5	z_{RHS} =56	

TABLE 2.1.4

6. Continue the table until all $\mathbf{c_j - z_j}$ values are either zero or negative. The corresponding final $\mathbf{z_{RHS}}$ is the maximum value. Also find corresponding $\mathbf{x_1}$ and $\mathbf{x_2}$

NOTE:

1) θ is always positive. In case it is coming as negative or undefined, leave the slot in the table blank.
2) Don't convert fractions into decimals. Final answers are natural numbers.

Problem 2.2. Maximise

$$6x_1 + 5x_2$$

with the constraints

$$x_1 + x_2 \leq 5$$
$$3x_1 + 2x_2 \leq 12$$
$$\text{where } x_1, x_2 \geq 0$$

using the tabular method. Find the corresponding values of x_1 & x_2.